# Pattern avoidance and smoothness of closures for orbits of a symmetric subgroup in the flag variety

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A R T I C L E   I N F O

A B S T R A C T

We give a pattern avoidance criterion to classify the orbits of $\mathrm{Sp}(p, \mathbb{C}) \times \mathrm{Sp}(q, \mathbb{C})$ (resp. $\mathrm{GL}(n, \mathbb{C})$) on the flag variety of type $\mathsf{C}_{p+q}$ (resp. $\mathsf{D}_n$) with rationally smooth closure. We show that all such orbit closures fiber (with smooth fiber) over a smaller flag variety, and hence are in fact smooth. In addition we prove that the classification is insensitive to isogeny.

Suppose $G$ is a complex connected reductive algebraic group and let $\theta$ denote an involutive automorphism of $G$. Write $K$ for the fixed points of $\theta$, and $\mathcal{B}$ for variety of maximal solvable subalgebras of the Lie algebra $\mathfrak{g}$ of $G$. (Henceforth we call this variety simply the flag variety of $G$.) Then $K$ acts with finitely many orbits on $\mathcal{B}$ via the restriction of the adjoint action (e.g. [Mat79]).

Since we have assumed the ground field is $\mathbb{C}$, $\theta$ arises as the complexification of a Cartan involution for a real form $G_{\mathbb{R}}$ of $G$. The localization theory of Beilinson–Bernstein relates the geometry of $K$ orbits on $\mathcal{B}$ with the category of Harish-Chandra modules for $G_{\mathbb{R}}$. Meanwhile, as a special case, one can consider the setting where $G_{\mathbb{R}}$ is itself a complex Lie group. In this case $G = G_{\mathbb{R}} \times G_{\mathbb{R}}$, $\theta$ is the involution that interchanges the two factors, $K$ is the diagonal copy of $G_{\mathbb{R}}$, and the Weyl group of $G_{\mathbb{R}}$ parametrizes the orbits of $K$ on $\mathcal{B}$ (which now is two copies of the flag variety $\mathcal{B}_{\circ}$ for $G_{\mathbb{R}}$). Intersecting such an orbit with (say) the left copy $\mathcal{B}_{\circ}$ gives an orbit of a Borel subgroup on $\mathcal{B}_{\circ}$, that is a Schubert cell. This process preserves the fine structure of the singularities of the closures of each kind of orbit, and is the geometric underpinning of the equivalence of categories (essentially) between category $\mathcal{O}$ and a suitable category of Harish-Chandra modules for $G_{\mathbb{R}}$ (e.g. [BorBry85]).

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Thus, roughly speaking, any question which one can ask about Schubert varieties can also be posed in the greater generality of $K$ orbits on $\mathcal{B}$, and any relation of the geometry of the former with category $\mathcal{O}$ can potentially be translated into a relation of the latter with the category of Harish-Chandra modules. Some of the deepest results in this direction are due to Lusztig and Vogan [LusVog83] and Vogan [Vog79,Vog83], which when taken together give an algorithm to compute the local intersection homology (with coefficients in any irreducible local system) of any orbit closure. When $G_{\mathbb{R}}$ is a complex group, the algorithm is equivalent to that of [KazLus79] for Schubert varieties.

Our interest here is determining when the closures of $K$ orbits on $\mathcal{B}$ are smooth, or more generally rationally smooth. Since the latter condition is equivalent to a condition on local intersection homology (see the discussion of [KazLus79, Appendix]), the question of whether a particular orbit has rationally smooth closure can be answered using the algorithm of [Vog79]. But it is desirable to have a "closed form" of the answer. For instance, in the case that $G_{\mathbb{R}}$ is complex (or, equivalently, the case of Schubert varieties), a closed form answer for smoothness or rational smoothness has been obtained in terms of a kind of pattern avoidance for Weyl group elements.[2]

In the setting of $K$ orbits on $\mathcal{B}$, new phenomena appear that are not present in the case of Schubert varieties. The most obvious difference is that while the definition of Schubert varieties is independent of the isogeny class of $G$, symmetric subgroups (and their orbits on $\mathcal{B}$) do indeed depend on isogeny. A typical complication in the latter case may be visualized as follows. Suppose $G$ is simply connected but not adjoint, $\overline{G} \neq G$ is isogenous to $G$, and $\overline{K}$ and $K$ are respective symmetric subgroups with the same Lie algebra. Then it frequently happens that $\overline{K}$ is disconnected (while $K$ is connected).[3] Thus there may be two distinct orbits $Q_1$ and $Q_2$ for $K$ on $\mathcal{B}$ whose union forms a single orbit $Q$ for $\overline{K}$. Schematically one encounters pictures as follows.
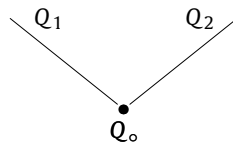


Here $Q_\circ$ is a closed $K$ orbit (which is also a $\overline{K}$ orbit) which appears in the closure of both $K$ orbits $Q_1$ and $Q_2$. The picture indicates that the closure of the $K$ orbit $Q_1$ (or $Q_2$) is smooth at $Q_\circ$. But the $\overline{K}$ orbit $Q$, the union of $Q_1$ and $Q_2$, has closure which isn't smooth (or even rationally smooth) at $Q_\circ$. Moreover, this particular example can "propagate" in higher rank schematically as follows.
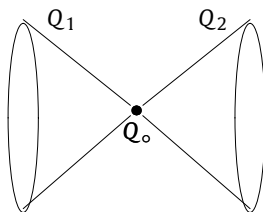


This time the closure of the $K$ orbit $Q_1$ (or $Q_2$) is no longer smooth at $Q_\circ$ (but is rationally smooth), but once again the $\overline{K}$ orbit $Q$ has closure which is not smooth (nor rationally smooth) at $Q_\circ$. Of course none of this is conceptually very complicated, but since, roughly speaking, pattern avoidance

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[2] There is an extensive literature of geometric and combinatorial results relating pattern avoidance in Weyl groups to singularities of Schubert varieties. We do not attempt to recount the history of these results in detail here. See [Bil98], [BilLak00, Chapter 8] or [BilPos05] and the extensive references therein.

[3] We remark that the notion of isogeny we are considering here differs from isogeny of the corresponding real forms. More precisely, if $G_{\mathbb{R}}$ and $\overline{G}_{\mathbb{R}}$ are the real forms corresponding to symmetric subgroups $K$ and $\overline{K}$ of groups $G$ and $\overline{G}$ with $G$ simply connected and $\overline{G}$ a quotient of $G$ by a central subgroup, then of course it does not follow that $\overline{G}_{\mathbb{R}}$ is a quotient of $G_{\mathbb{R}}$ by a central subgroup. (In other words, the relevant central subgroup of $G$ need not be defined over $\mathbb{R}$.) For the results of this paper, the latter notion of isogeny of real forms is not interesting and will not be considered.

results are predicated on failure of smoothness propagating uniformly to higher rank, these examples suggest that such results are potentially more subtle in the case of $K$ orbits in ways that are not seen for Schubert varieties. This is made clear by Example 5.1.

In this paper, we are interesting in understanding nice isogeny-independent pattern avoidance results. For instance, [McG07] answers the questions of smoothness and rational smoothness for the closures of orbits of $K = \mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C})$ on the flag variety for $G = \mathrm{GL}(p+q, \mathbb{C})$. This is the setting arising from the real form $G_{\mathbb{R}} = \mathrm{U}(p, q)$ of $G$. After reviewing some preliminaries in Section 1, we recall the results of [McG07] in Section 2 and then prove that they are, in a suitable sense, insensitive to isogeny.

In the remainder of the paper, we go on to study other classical groups outside of type A. In general, as remarked above, the situation potentially depends crucially on issues related to isogeny. But, perhaps surprisingly, we find that isogeny essentially plays no role if $G$ is of type C (resp. type D) and the Lie algebra of $K$ is $\mathfrak{sp}(p, \mathbb{C}) \oplus \mathfrak{sp}(q, \mathbb{C})$ (resp. $\mathfrak{gl}(n, \mathbb{C})$). This setting includes the cases arising from the real groups $\mathrm{Sp}(p, q)$ and $\mathrm{SO}^*(2n)$. The main results are Theorems 3.2 and 4.2. They completely settle the question of rational smoothness (which, in the end, turns out to be equivalent to smoothness) for $K$ orbits closures in these cases. The statements are formulated in terms of a remarkably simple pattern avoidance criterion resembling the one discovered in [McG07]. (Roughly speaking there are always only seven "bad" patterns to avoid.)

## 1. Preliminaries

In particular examples below, we will need a detailed description of the closure order of $K$ orbits on $\mathcal{B}$. We begin by recalling a few features of the general case. These are due to Matsuki [Mat79], Matsuki and Oshima [MatOsh88], Lusztig and Vogan [LusVog83], and are given a full exposition in Richardson and Springer [RicSpr90].

Following the terminology of [RicSpr90, Section 5.1], we first recall the weak closure order on $K \backslash \mathcal{B}$. Fix, once and for all, a choice of $\theta$-stable Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$, and a choice of positive roots $\Delta^+$ in the full root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$. Write $B$ for the corresponding Borel subgroup of $G$. For a simple root $\alpha \in \Delta^+$, let $\mathcal{P}_\alpha$ denote the variety of parabolic subalgebras of type $\alpha$ (i.e. those conjugate to the one with roots $\Delta^+ \cup \{-\alpha\}$). Write $\pi_\alpha$ for the projection of $\mathcal{B}$ to $\mathcal{P}_\alpha$, and define

$$s_\alpha \cdot \mathcal{O} := \text{the (unique) dense } K \text{ orbit in } \pi_\alpha^{-1}\big(\pi_\alpha(\mathcal{O})\big). \tag{1}$$

The weak closure order is generated by relations $\mathcal{O} < s_\alpha \cdot \mathcal{O}$ whenever $\dim(s_\alpha \cdot \mathcal{O}) = \dim(\mathcal{O}) + 1$. In this case, for $\mathcal{O}' = s_\alpha \cdot \mathcal{O}$, we write

$$\mathcal{O} \xrightarrow{\alpha} \mathcal{O}'.$$

Then $\mathcal{O} \subset \overline{\mathcal{O}'}$, but these relations do not generate the full closure order. To obtain all closure relations, we must recursively apply the following procedure (implicit in [RicSpr90, Theorem 7.11(vii)], for instance). Whenever a codimension one subdiagram of the form



$$\tag{2}$$

is encountered, it must be completed to

$$
\begin{array}{ccc}
& \mathcal{O}_1 & \\
\alpha \nearrow & & \nwarrow \\
\mathcal{O}_2 & & \mathcal{O}_3 \\
\nwarrow & & \nearrow \alpha \\
& \mathcal{O}_4 &
\end{array}
\tag{3}
$$

New edges added in this way are represented by dashed lines in the figures appearing in Section 5. Note that this operation must be applied recursively, and thus the solid unlabeled edge in the original diagram (2) may be dashed as the recursion unfolds.

We next recall a definition from [RicSpr90, 4.7]. Given $w \in W = W(\mathfrak{h}, \mathfrak{g})$, fix a reduced expression $w = s_{\alpha_k} \cdots s_{\alpha_2} s_{\alpha_1}$. For $\mathcal{O} \in K \backslash \mathcal{B}$, define

$$
w \cdot \mathcal{O} = s_{\alpha_k} \cdot \left( \cdots s_{\alpha_2} \cdot (s_{\alpha_1} \cdot \mathcal{O}) \right).
\tag{4}
$$

It is easy to see this is well-defined independent of the choice of reduced expression.

The following is our main tool to detect failure of rational smoothness, and is a special case of the results of Springer [Spr92].

**Theorem 1.1.** *Suppose $\mathfrak{g}$ contains a Cartan subalgebra which is fixed pointwise by $\theta$. Fix $\mathcal{O} \in K \backslash \mathcal{B}$ and a closed orbit $\mathcal{O}_{cl} \in K \backslash \mathcal{O}$. Consider*

$$
S(\mathcal{O}, \mathcal{O}_{cl}) = \{\alpha \in \Delta^+ \mid s_\alpha \cdot \mathcal{O}_{cl} \neq \mathcal{O}_{cl} \text{ and } s_\alpha \cdot \mathcal{O}_{cl} \subset \overline{\mathcal{O}}\}.
$$

*If*

$$
\#S(\mathcal{O}, \mathcal{O}_{cl}) > \dim(\mathcal{O}) - \dim(\mathcal{O}_{cl}),
\tag{5}
$$

*then the closure of $\mathcal{O}$ is not rationally smooth.*

The necessary condition for rational smoothness furnished by Theorem 1.1 is a priori rather weak; for example, the analogous necessary condition for rational smoothness of complex Schubert varieties is far from sufficient, even in low rank. We will see below, however, that this condition is both necessary and sufficient for rational smoothness in all the cases that we consider. (Even though we make no use of them, we mention that a number of other powerful techniques exist for detecting (rational) smoothness; see [Bri99,Bri00,CarKut03], for example, and the references therein.)

## 2. U($p, q$)

We specialize to the setting of [McG07] before returning to questions of isogeny at the end of this section. Fix integers $p + q = n$, and a signature $(p, q)$ Hermitian form $\langle \cdot, \cdot \rangle$ on an $n$-dimensional complex vector space $V_{\mathbb{C}}$. Let $G = \text{GL}(V_{\mathbb{C}})$ (the invertible complex linear endomorphisms of $V_{\mathbb{C}}$ with determinant one) and let $G_{\mathbb{R}}$ denote the subgroup of $G$ preserving $\langle \cdot, \cdot \rangle$. Write $\theta$ for the complexification of a Cartan involution of $G_{\mathbb{R}}$ and set $K = G^\theta$. Then $K \simeq \text{GL}(p, \mathbb{C}) \times \text{GL}(q, \mathbb{C})$. Choose coordinates so that $\Delta^+ = \{e_i - e_j \mid i < j\}$.

In the present setting, the twisted involutions of [RicSpr90, Section 3] parametrizing $K \backslash \mathcal{B}$ amount to *involutions of $S_{p+q}$ with signed fixed points of signature $(p, q)$*; that is, involutions in the symmetric group $S_{p+q}$ whose fixed points are labeled with signs (either $+$ or $-$) so that half the number of non-fixed points plus the number of $+$ signs is exactly $p$ (or, equivalently, half the number of non-fixed points plus the number of $-$ signs is $q$). The parametrization is arranged to have the following feature.

Suppose $\mathcal{O}$ is parametrized by an involution with signed fixed points whose underlying involution in the symmetric group is $\sigma$. Then there is a representative $\mathfrak{b}_{\mathcal{O}} = \mathfrak{h}_{\mathcal{O}} \oplus \mathfrak{n}_{\mathcal{O}}$ (with $\mathfrak{h}_{\mathcal{O}}$ $\theta$-stable) with the following property. Write $\Delta_{\mathcal{O}}^+$ for the roots of $\mathfrak{h}_{\mathcal{O}}$ in $\mathfrak{g}$. There is a unique inner automorphism of $\mathfrak{g}$ carrying $\mathfrak{h}$ to $\mathfrak{h}_{\mathcal{O}}$ and $\Delta^+$ to $\Delta_{\mathcal{O}}^+$. Using it we may transport the action of $\theta$ on $\Delta_{\mathcal{O}}^+$ to our fixed system $\Delta^+$. Once this is done, the key property is that

$$\theta(e_i - e_j) = e_{\sigma(i)} - e_{\sigma(j)}. \tag{6}$$

Let $\Sigma_{\pm}(p, q)$ denote the set of signed involutions of $S_{p+q}$ with signature $(p, q)$ as in the previous paragraph. For the purposes of formulating pattern avoidance results, we introduce the following notation. An element $\gamma \in \Sigma_{\pm}(p, q)$ will be identified with an $n$-tuple $(c_1, \ldots, c_n)$, with each $c_i$ either a natural number, a $+$, or a $-$ such that: every natural number occurs exactly twice; and the number of distinct natural numbers plus the number of $+$ entries is exactly $p$. For later use in Definition 2.1 below, we say that two such strings are *equivalent* if they have the same signs in the same position and pairs of equal numbers in the same positions. (So $11 + - 22$ is equivalent to $22 + - 33$ and $55 + - 22$, for instance.) In any event, the correspondence between equivalence classes of such strings and involutions with signed fixed points is clear: pairs of equal natural numbers $c_i = c_j$ in the string correspond to indices $i$ and $j$ interchanged by the involution, and a sign $c_i$ in the string corresponds to a label of the fixed points $i$ of the involution. We will generally not distinguish between elements of $\Sigma_{\pm}(p, q)$ and (equivalence classes of) such strings (which, we remark, are called "clans" in [Yam97]).

We now turn to an explicit description of the closure order. (It may be helpful to refer to Fig. 1 when reading the discussion below.) Fix $\gamma = (c_1, \ldots, c_n) \in \Sigma_{\pm}(p, q)$, a simple root $\alpha = e_i - e_{i+1} \in \Delta^+$, and recall the action (described before (6)) of $\theta$ on $\alpha$ determined by $\mathcal{O}_{\gamma}$. Then

$$\dim(s_\alpha \cdot \mathcal{O}_\gamma) = 1 + \dim(\mathcal{O}_\gamma)$$

if and only if one of the following conditions hold (using terminology as in [RicSpr90, 4.3], for instance):

(a) $\alpha$ is complex (i.e. $\theta(\alpha) \neq \pm\alpha$) and $\theta\alpha \in \Delta^+$;
(b) $\alpha$ is imaginary (i.e. $\theta(\alpha) = \alpha$) and noncompact.

Using (6) for the former condition, and a calculation in U(1, 1) for the latter, these may be formulated as the following conditions on positions $i$ and $i + 1$ in $\gamma$:

(a) $c_i$ and $c_{i+1}$ are unequal natural numbers such that $j < k$ where $c_i = c_j$ and $c_{i+1} = c_k$ (and $j \neq i$ and $k \neq i + 1$);
(a′) $c_i$ is a sign, $c_{i+1}$ is a number and the entry $c_k$ with $c_k = c_{i+1}$ ($k \neq i + 1$) satisfies $i < k$;
(a″) $c_i$ is a number, $c_{i+1}$ is a sign, and the other entry $c_k$ with $c_k = c_i$ ($k \neq i$) satisfies $k < i + 1$;
(b) $c_i$ and $c_{i+1}$ are opposite signs.

Thus if $i = 2$, then $\gamma_1 = (1, 1, 2, 2)$ satisfies the first condition above, $\gamma_2 = (-, +, 1, 1)$ satisfies the second, $\gamma_3 = (1, 1, -, +)$ the third, $\gamma_4 = (1, +, -, 1)$ the fourth, and $(1, 2, 1, 2)$ or $(1, +, 1, -)$ satisfies none of them. In each case the dense $K$ orbit in $s_\alpha \cdot \mathcal{O}_\gamma$ is parametrized by $\gamma' = (c_1', \ldots, c_n') \in \Sigma_{\pm}(p, q)$ which differs from $\gamma$ only in the $i$ and $i + 1$ entries in each respective case as follows:

(a) $c_i' = c_{i+1}$ and $c_{i+1}' = c_i$;
(b) $c_i' = c_{i+1}'$ is a natural number.

So for the examples listed above, we have $\gamma_1' = (1, 2, 1, 2)$, $\gamma_2' = (-, 1, +, 1)$, and $\gamma_3' = (1, -, 1, +)$, and $\gamma_4' = (1, 2, 2, 1)$.

The previous paragraph thus gives a complete description of the weak closure order. After applying the recursive procedure given above, one obtains an explicit description of the full closure order.

In particular, all of the following operations move from one element $\gamma \in \Sigma_\pm(p,q)$ to a higher one in the order: replace a pair of (not necessarily adjacent) opposite signs by a pair of equal numbers; or interchange a number with a sign (again not necessarily adjacent to it) so as to move the number farther away from its equal mate in the string (and on the same side); or interchange a pair $a$, $b$ of equal numbers with $a$ to the left of $b$ provided that the mate of $a$ lies to the left of the mate of $b$. Thus (the orbit corresponding to) $(1, +, 1, -)$ lies below $(1, 2, 1, 2)$ and $(1, +, -, 1)$, while $(1, 2, 1, 3, 2, 3)$ lies below $(1, 3, 1, 2, 2, 3)$ but not below $(1, 3, 1, 3, 2, 2)$. In particular, the closed orbits are parametrized by elements of $\Sigma_\pm(p,q)$ consisting only of signs, while the open orbit is parametrized by $(1, 2, \ldots, 2q, +, \ldots, +, 2q-1, 2q, 2q-3, 2q-2, \ldots, 1, 2)$, with $2p - 2q$ plus signs, if $p > q$.

By similar considerations, one quickly deduces the following dimension formula (as in [Yam97, Section 2.3], for instance) for the orbit $\mathcal{O}_\gamma$ parametrized by $\gamma = (c_1, \ldots, c_n)$. Let

$$l(\gamma) = \sum_{c_i = c_j \in \mathbb{N}, \, i < j} \left( j - i - \#\{k \in \mathbb{N} \mid c_s = c_t = k \text{ for some } s < i < t < j\} \right). \tag{7}$$

Then

$$\dim(\mathcal{O}_\gamma) = d(K) + l(\gamma),$$

where $d(K)$ is the dimension of the flag variety for $K$, namely $\frac{1}{2}(p(p-1) + q(q-1))$.

**Definition 2.1.** We say that an involution with signed fixed points $(c_1, \ldots, c_n) \in \Sigma_\pm(p,q)$ includes the pattern $(d_1, \ldots, d_m)$ if there are indices $i_1 < \cdots < i_m$ so that the (possibly shorter) string $(c_{i_1}, \ldots, c_{i_m})$ is equivalent to $(d_1, \ldots, d_m)$. We say that $\gamma$ avoids $(d_1, \ldots, d_m)$ if it does not include it. For instance, $(1, 1, 2, +, 3, 2, -, 3)$: contains the pattern $(1, 1, 2, 2)$ (by considering $i_1 = 1$, $i_2 = 2$, $i_3 = 3$, and $i_4 = 6$), contains $(1, 2, 1, 2)$ (by considering $i_1 = 3$, $i_2 = 5$, $i_3 = 6$, and $i_4 = 8$), and contains $(1, +, 1, -)$ (by considering $i_1 = 3$, $i_2 = 4$, $i_3 = 6$, and $i_4 = 7$); but avoids $(1, +, -, 1)$, $(1, +, +, 1)$, and $(1, 2, 2, 1)$.

Here is the main result from [McG07].

**Theorem 2.2.** *Fix $\gamma \in \Sigma_\pm(p,q)$, an involution in $S_{p+q}$ with signed fixed points of signature $(p,q)$ (as defined above). If $\gamma$ includes one of the patterns $(1, +, -, 1)$, $(1, -, +, 1)$, $(1, 2, 1, 2)$, $(1, +, 2, 2, 1)$, $(1, -, 2, 2, 1)$, $(1, 2, 2, +, 1)$, or $(1, 2, 2, -1)$ then the closure of $\mathcal{O}_\gamma$ is not rationally smooth. In all other cases, $\mathcal{O}_\gamma$ has smooth closure. In particular, an orbit has smooth closure if and only if it has rationally smooth closure, or if and only if the condition of Theorem* 1.1 *fails for some closed orbit below it.*

Next we turn to issues of isogeny. First, we switch notation and consider the simply connected simple group $G = \mathrm{SL}(n, \mathbb{C})$. Since the center of $\mathrm{GL}(n, \mathbb{C})$ acts trivially on $\mathcal{B}$, the above discussion applies without change for symmetric subgroup $K = \mathrm{S}(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}))$ of $G$. Now let $\overline{G}$ be a quotient of $G$ by a subgroup $F < \mathbb{Z}/n$ of the center of $G$. Let $\overline{K}$ be the symmetric subgroup of $\overline{G}$ with Lie algebra $\mathfrak{s}(\mathfrak{gl}(p, \mathbb{C}) \oplus \mathfrak{gl}(q, \mathbb{C}))$. Then the orbits of $\overline{K}$ on $\mathcal{B}$ coincide with those of $K$ *except* in one case: if $n = 2m$, $p = q = m$, and $F$ contains the index $m$ subgroup of $\mathbb{Z}/2m$. In this case, $\overline{K}$ has two connected components, and indeed $\overline{K}$ orbits can be disconnected. More precisely, $\overline{K}$ orbits are parametrized by $\Sigma_\pm(p,q)$ modulo the equivalence relation generated by $\gamma \sim -\gamma$, where $-\gamma$ is obtained from $\gamma$ by reversing all signs of fixed points. (So, for instance, $(1, +, -, 1)$ is equivalent to $(1, -, +1)$, but $(1, 2, 1, 2)$ is equivalent to only itself.) Equivalence classes thus have either one or two elements. In general, if $\overline{\gamma}$ denotes such a class, the $\overline{K}$ orbit $\mathcal{O}_{\overline{\gamma}}$ parametrized by $\overline{\gamma}$ breaks into $K$ orbits as

$$\mathcal{O}_{\overline{\gamma}} = \mathcal{O}_\gamma \cup \mathcal{O}_{-\gamma} \tag{8}$$

which is disconnected if $\gamma \neq -\gamma$. So the subtleties alluded to in the introduction could potentially come into play. In fact, they do not, and one way to see this is to examine the proof of Theorem 2.2.

It consists of two steps: first proving that if $\gamma$ contains one of the three indicated patterns, then it is not rationally smooth; and, second, proving that the remaining orbits are smooth. The first is done by finding a suitable closed orbit $\mathcal{O}_{cl}$ in the closure of $\mathcal{O}_\gamma$ so that Theorem 1.1 applies. It turns out that the identical argument can be carried out for $\overline{K}$ orbits; that is, roughly speaking, Springer's criterion is insensitive to isogeny in this case. (This need *not* always be true—see Example 5.1.) The second step can also be carried out in an analogous manner which, once again, is a special feature of this case. In the end, one concludes *Theorem 2.2 holds for the orbits of any symmetric subgroup $\overline{K}$ of $\overline{G}$* (*isogenous to* SL$(n, \mathbb{C})$) *with Lie algebra* $\mathfrak{s}(\mathfrak{gl}(p, \mathbb{C}) \oplus \mathfrak{gl}(q, \mathbb{C}))$.

We conclude this section by recalling (as in [RicSpr90, Section 2]) the natural action of $W = W(\mathfrak{h}, \mathfrak{g})$ on the set of twisted involutions parametrizing $K$ orbits on $\mathcal{B}$. (We will need this for formulating some results in Section 4.) On the level of $\Sigma_\pm(p, q)$ we write $w \times \gamma$ for the action of $w$ on $\gamma$. Explicitly, it amounts to the obvious action of the symmetric group: the $w(i)$th component of (the string parametrizing) $w \times \gamma$ is simply the $i$th component of (the string parametrizing) $\gamma$.

## 3. Sp$(p, q)$

We now turn our attention to $G_\mathbb{R} = \text{Sp}(p, q)$, a real form of $G = \text{Sp}(2n, \mathbb{C})$, again deferring isogeny questions to the end of this section. We start with a brief outline of our strategy. As the definition recalled below makes evident, $G_\mathbb{R}$ is a subgroup of $G'_\mathbb{R} = \text{U}(2p, 2q)$. In this section, we will try (as much as possible) to reduce the study of $K$ orbits on $\mathcal{B}$, the flag variety for $G$, to corresponding results in the previous section for $K' \simeq \text{GL}(2p, \mathbb{C}) \times \text{GL}(2q, \mathbb{C})$ orbits on $\mathcal{B}'$, the flag variety for $\text{GL}(2p + 2q, \mathbb{C})$. More precisely, $\mathcal{B}$ naturally includes into $\mathcal{B}'$ and (for appropriate choices of the Cartan involutions in question) a $K'$ orbit on $\mathcal{B}'$ either meets $\mathcal{B}$ in a single $K$ orbit, or else not at all. Thus the orbits of $K$ on $\mathcal{B}$ are parametrized by a subset of the involutions with signed fixed points $\Sigma_\pm(2p, 2q)$ introduced above. In fact, we quickly check below that the closure order on $K$ orbits on $\mathcal{B}$ is simply the appropriate restriction of the closure order of $K'$ orbits on $\mathcal{B}'$. Thus the simplest result that one could hope for is this: the closure of an orbit for $K$ on $\mathcal{B}$ has (rationally) smooth closure if and only if the corresponding $K'$ orbit $\mathcal{B}'$ which it meets has (rationally) smooth closure (and recall these latter orbits have been classified in Theorem 2.2). This is in fact the content of Theorem 3.2 below, apart from an easily stated exception treated in Lemma 3.1.

It is worth remarking that since (up to issues of isogeny) any classical real group outside of type A is a subgroup of an appropriate $\text{U}(p, q)$, one could attempt to mimic the strategy of the previous paragraph for any such group. In the next section, we do so for $\text{SO}^*(2n)$ and obtain similar results. But for the other classical groups, complications arise; see Example 5.1.

We return to the details of the case of $\text{Sp}(p, q)$. Let $\mathbb{H}$ denote the quaternions equipped with the standard bar operation $\overline{a + bi + cj + dk} = a - bi - cj - dk$. Embed $\mathbb{C}$ in $\mathbb{H}$, as usual, as elements of the form $a + bi$ and write the corresponding isomorphism $\mathbb{H} \simeq \mathbb{C}^2$ as $z = \text{A}(z) + \text{B}(z)j$. Let $\langle \cdot, \cdot \rangle$ denote a signature $(p, q)$ sesquilinear form on an $n$-dimensional quaternionic (left) vector space $V_\mathbb{H}$. Define

$$\langle \cdot, \cdot \rangle' = \text{A} \circ \langle \cdot, \cdot \rangle$$

and

$$\langle \cdot, \cdot \rangle'' = \text{B} \circ \langle \cdot, \cdot \rangle.$$

Then $\langle \cdot, \cdot \rangle'$ is a nondegenerate Hermitian form of signature $(2p, 2q)$ on the underlying $2n$-dimensional complex vector space $V_\mathbb{C}$ and $\langle \cdot, \cdot \rangle''$ is a nondegenerate alternating form on $V_\mathbb{C}$.

Let $G_\mathbb{R} = \text{Sp}(p, q)$ denote the subgroup of $\text{GL}(V_\mathbb{H})$ (the group of invertible left $\mathbb{H}$-linear endomorphisms of $V_\mathbb{H}$) preserving $\langle \cdot, \cdot \rangle$ and view, as we may, $G_\mathbb{R}$ as subgroup of $\text{GL}(V_\mathbb{C})$. Let $G'_\mathbb{R}$ denote the subgroup of $G' := \text{GL}(V_\mathbb{C})$ preserving $\langle \cdot, \cdot \rangle'$, and finally let $G$ denote the subgroup of $\text{GL}(V_\mathbb{C})$ preserving $\langle \cdot, \cdot \rangle''$. Then $G_\mathbb{R}$ is a real form of $G \simeq \text{Sp}(2n, \mathbb{C})$, $G'_\mathbb{R} \simeq \text{U}(2p, 2q)$, and

$$G_\mathbb{R} = G'_\mathbb{R} \cap G.$$

We adopt the notation of Section 2 for $G'_{\mathbb{R}}$, adding a prime everywhere as appropriate. Any Cartan involution for $G'_{\mathbb{R}}$ restricts to one for $G_{\mathbb{R}}$. If we write $\theta'$ for the corresponding complexified involution of $G'$ and $\theta$ for its restriction to $G$, we naturally have

$$\mathrm{Sp}(2p, \mathbb{C}) \times \mathrm{Sp}(2q, \mathbb{C}) \simeq K := G^\theta < K' := (G')^{\theta'} \simeq \mathrm{GL}(2p, \mathbb{C}) \times \mathrm{GL}(2q, \mathbb{C}).$$

Recall the natural inclusion of the flag variety $\mathcal{B}$ for $G$ into $\mathcal{B}'$, the flag variety for $G'$. By the remarks above, in order to classify $K$ orbits on $\mathcal{B}$ it suffices to determine which orbits $\mathcal{O}'$ meet $\mathcal{B}$ nontrivially. If $\gamma' = (c'_1, \ldots, c'_{2n}) \in \Sigma_\pm(2p, 2q)$ (with notation as in the last section), we say that it is symmetric if:

(i) If the entry $c'_i$ is a sign, then $c'_{2n+1-i}$ is the same sign.
(ii) If $c'_i = c'_j$ are natural numbers, then $j \neq 2n + 1 - i$ and $c'_{2n+1-i} = c'_{2n+1-j}$.

We write $\Sigma_\pm^{\mathrm{sym}}(2p, 2q)$ for the subset of symmetric elements in $\Sigma_\pm(2p, 2q)$. Then the $K'$ orbit $\mathcal{O}'_{\gamma'}$ meets $\mathcal{B}$ if and only if $\gamma'$ is symmetric ([MatOsh88], [Yam97, Section 4.3]), and thus $\Sigma_\pm^{\mathrm{sym}}(2p, 2q)$ parametrizes $K$ orbits on $\mathcal{B}$. Our next task is to describe the closure order explicitly.

Fix, as in Section 2, a $\theta'$-stable Cartan subalgebra $\mathfrak{h}'$, a choice of positive roots, and (in appropriate coordinates) write $(\Delta')^+ = \{e'_i - e'_j \mid i < j\}$. Then $\mathfrak{h} := \mathfrak{g} \cap \mathfrak{h}'$ is a $\theta$-stable Cartan subalgebra of $\mathfrak{g}$. Restriction defines a positive system of roots of $\mathfrak{h}$ in $\mathfrak{g}$, $\Delta = \{2e_i \mid 1 \leqslant i \leqslant n\} \cup \{e_i \pm e_j, \ 1 \leqslant i < j \leqslant n\}$. Let $B$ denote the corresponding Borel subgroup of $G$. Fix a simple root $\alpha$ in $\Delta^+$ and let $S'(\alpha)$ the set of roots in $(\Delta')^+$ which restrict to $\alpha$. Concretely, if $\alpha = e_i - e_{i+1}$, then $S'(\alpha) = \{\alpha', \alpha''\}$ with $\alpha' = e'_i - e'_{i+1}$ and $\alpha'' = e'_{2n-i} - e'_{2n-i+1}$; and if $\alpha = 2e_1$, then $S'(\alpha) = \{\alpha'\}$ where $\alpha' = e'_n - e'_{n+1}$.

Given a simple root $\alpha \in \Delta^+$ and $\mathcal{O}$ in $K \backslash \mathcal{B}$, define $s_\alpha \cdot \mathcal{O}$ as in (1). For $\beta \in (\Delta')^+$ and $\mathcal{O}'$ in $K' \backslash \mathcal{O}'$, define $s'_\beta \cdot \mathcal{O}'$ similarly. The parametrization of $K$ and $K'$ orbits satisfies the following key geometric compatibility condition (which follows easily from unraveling the definitions). Given any symmetric $\gamma$, let $\mathcal{O}_\gamma$ denote the corresponding $K$ orbit on $\mathcal{B}$ and $\mathcal{O}'_\gamma$ the corresponding $K'$ orbits on $\mathcal{B}'$. Then

$$\dim(s_\alpha \cdot \mathcal{O}_\gamma) = 1 + \dim(\mathcal{O}_\gamma) \tag{9}$$

if and only if

$$\dim(s'_\beta \cdot \mathcal{O}'_\gamma) = 1 + \dim(\mathcal{O}'_\gamma) \tag{10}$$

for some (equivalently, any) root $\beta \in S'(\alpha)$. Moreover, in this case, the dense $K'$ orbit in

$$\left[ \prod_{\beta \in S'(\alpha)} s'_\beta \right] \cdot (\mathcal{O}'_\gamma) \tag{11}$$

intersected with $\mathcal{B}$ is the dense $K$ orbit in

$$\pi_\alpha^{-1}\big(\pi_\alpha(\mathcal{O}_\gamma)\big). \tag{12}$$

Note that if the product in (11) has more than one term, the order of the terms does not matter since, in this case, the roots in $S'(\alpha)$ are orthogonal.

Using these facts, it is a simple matter to write down the weak closure order on the level of $\Sigma_\pm^{\mathrm{sym}}(2p, 2q)$ from the description of the weak closure order described in the previous section. (This is done in [Yam97, 4.4], for instance.) By examining the recursive procedure to generate the full closure order from the weak order, one concludes that the closure order of $K$ orbits on the level

$\Sigma_{\pm}^{\mathrm{sym}}(2p, 2q)$ is simply the restriction of the order on $\Sigma_{\pm}(2p, 2q)$ given in the previous section. The dimension of the orbit parametrized by $\gamma \in \Sigma_{\pm}^{\mathrm{sym}}(2p, 2q)$ is also easy to read off, and is given by

$$\dim(\mathcal{O}_\gamma) = d(K) + (1/2)\big(l(\gamma) + \{t \in \mathbb{N} \mid c_s = c_t \in \mathbb{N} \text{ with } s \leqslant n < t \leqslant 2n + 1 - s\}\big)$$

where $d(K)$ is the dimension of the flag variety for $K$, namely $p^2 + q^2$, and $l(\gamma)$ is as in (7). In particular, closed orbits are once again parametrized by elements consisting only of signs, while the open orbit is parametrized by

$$\gamma_\circ(p, q) := (1, 2, \ldots, 2q, +, \ldots, +, 2q - 1, 2q, 2q - 3, 2q - 2, \ldots, 1, 2) \tag{13}$$

with $p - q$ plus signs if $p \geqslant q$, and similarly if $q \geqslant p$. See Fig. 2 for a detailed example.

We need some notation for the next result. If $\gamma_1 = (c_1, \ldots, c_k)$ represents an involution with signed fixed points, then let $\gamma_1^r$ be the string with the coordinates of $\gamma$ in reverse order but with each pair of equal numbers changed to a different pair of numbers. For example, if $\gamma_1 = (1, 2, +, 1, 2, -)$, then $\gamma_1^r = (-, 3, 4, +, 3, 4)$. For any such $\gamma_1$, the concatenation $\gamma := (\gamma_1, \gamma_1^r)$ is symmetric, so may be viewed as an element of $\Sigma_{\pm}^{\mathrm{sym}}(2p, 2q)$.

**Lemma 3.1.** *Fix $\gamma \in \Sigma_{\pm}^{\mathrm{sym}}(2p, 2q)$. Suppose that there are integers $p' + r = p$ and $q' + s = q$ so that $\gamma$ can be written as the concatenation*

$$\big(\gamma_1, \gamma_\circ(p', q'), \gamma_1^r\big)$$

*with $\gamma_1 \in \Sigma_{\pm}(r, s)$ such that $\gamma_1$ avoids the bad patterns of Theorem 2.2. Then the closure of $\mathcal{O}_\gamma$ is a fiber bundle with smooth fiber over a partial flag variety for $K$ and hence is smooth. The fiber is isomorphic to the product of the flag variety for $\mathrm{Sp}(2p' + 2q', \mathbb{C})$ and the closure of the $\mathrm{GL}(r, \mathbb{C}) \times \mathrm{GL}(s, \mathbb{C})$ orbit parametrized by $\gamma_1$ (as in Section 2).*

**Proof.** We start with a general observation. Suppose $\gamma$ is any element of $\Sigma_{\pm}(p, q)$ which can be written as a concatenation $(\gamma_1, \gamma_\circ, \gamma_1^r)$ where $\gamma_\circ \in \Sigma_{\pm}(p', q')$ and $\gamma_1 \in \Sigma_{\pm}^{\mathrm{sym}}(2r, 2s)$. Set $n' = p' + q'$, $n'' = r + s$. Then there is a $\theta$-stable parabolic subgroup $Q = LU$, unique up to $K$ conjugacy, containing $B$ with $L \simeq \mathrm{Sp}(n', \mathbb{C}) \times \mathrm{GL}(n'', \mathbb{C})$ and

$$K \cap L \simeq \mathrm{Sp}(p', \mathbb{C}) \times \mathrm{Sp}(q', \mathbb{C}) \times \mathrm{GL}(r, \mathbb{C}) \times \mathrm{GL}(s, \mathbb{C}).$$

The assumption that $\gamma = (\gamma_1, \gamma_\circ, \gamma_1^r)$ implies that the image of $\mathcal{O}_\gamma$ under projection $\pi_{\mathcal{P}}$ from $\mathcal{B} \simeq G/B$ to $\mathcal{P} \simeq G/Q$ is the closed $K$ orbit of the identity coset $eQ$ and that the fiber of the restriction of $\pi_{\mathcal{P}}$ to $\mathcal{O}_\gamma$ is a single orbit of $L \cap K$ on $\mathcal{B}_L$, the flag variety for $L$. The fiber is isomorphic to the product of the $\mathrm{Sp}(p', \mathbb{C}) \times \mathrm{Sp}(q', \mathbb{C})$ orbit parametrized by $\gamma_\circ$ with the orbit of $\mathrm{GL}(r, \mathbb{C}) \times \mathrm{GL}(s, \mathbb{C})$ parametrized by $\gamma_1$. Meanwhile the $K$ orbit of $eQ$ is closed, and hence isomorphic to a partial flag variety for $K$. The closure of $\mathcal{O}_\gamma$ is thus a fiber bundle over a partial flag variety for $K$ whose fiber is the closure of the product of the orbits parametrized by $\gamma_\circ$ and $\gamma_1$. If we impose the additional hypothesis that $\gamma_\circ = \gamma_\circ(p', q')$ and $\gamma_1$ avoids the patterns given in the lemma, then we conclude from Theorem 2.2 that the fiber is indeed smooth, and the remainder of the lemma follows.  $\square$

**Theorem 3.2.** *Fix $\gamma \in \Sigma_{\pm}^{\mathrm{sym}}(2p, 2q)$, a symmetric involution of $S_{2p+2q}$ with signed fixed point of signature $(2p, 2q)$ (as defined above). Suppose $\gamma$ is not of the form treated by Lemma 3.1 and, further, that $\gamma$ includes one of the bad patterns of Theorem 2.2. Then the closure of $\mathcal{O}_\gamma$ does not have rationally smooth closure.*

*In all other cases, the closure of $\mathcal{O}_\gamma$ is a fiber bundle with smooth fiber over a partial flag variety for $K$, and hence is smooth. In this case, there are integers $r + s + n' = p + q$ so that the fiber is isomorphic to the product of a flag variety for $\mathrm{Sp}(2n', \mathbb{C})$ and the closure of an orbit of $\mathrm{GL}(r, \mathbb{C}) \times \mathrm{GL}(s, \mathbb{C})$ on the flag variety for $\mathrm{GL}(r + s, \mathbb{C})$.*

*In particular, $\mathcal{O}_\gamma$ has rationally smooth closure if and only if it has smooth closure, or if and only if the condition of Theorem 1.1 fails for some closed orbit below it.*

**Proof.** If $\gamma$ is of the form treated by Lemma 3.1, there is nothing to prove. So suppose this is not the case and that $\gamma$ avoids the patterns given in the theorem. Then $\gamma$ takes the form $(\gamma_1, \gamma_1^r)$ with $\gamma_1 \in \Sigma_\pm(p, q)$ which avoids the same patterns. Thus Lemma 3.1 applies (with $p' = q' = 0$) to give the required assertion.

Suppose $\gamma$ contains one of the patterns of the theorem. Then there are various geometric ways to deduce the failure of the closure of $\mathcal{O}_\gamma$ to be rationally smooth by embedding the corresponding failure for the smaller rank group where, roughly speaking, the pattern resides. (The exceptions of Lemma 3.1 shows that some care is required.) A concise (and convenient) way to organize the geometric reduction is to use Theorem 1.1, and this is how we shall proceed.

In the present context a root $e_i - e_j$ with $i < j$ is noncompact imaginary for an orbit $\mathcal{O}_{cl}$ parametrized $cl$ if the $i$th and $j$th coordinates of $cl$ are opposite signs and the root $e_i + e_j$ is noncompact imaginary if the $i$th and $2n + 1 - j$th coordinates are opposite signs. (The root $2e_i$ is never noncompact imaginary.) Then if $\alpha = e_i - e_j$, $s_\alpha \cdot \mathcal{O}_{cl}$ is parametrized by the element obtained from $cl$ by replacing its $i$th, $j$th and $2n + 1 - j$th, $2n + 1 - i$th coordinates by different pairs of equal numbers; if instead $\alpha = e_i + e_j$, the new element is obtained by replacing the $i$th, $2n + 1 - j$th and $j$th, $2n + 1 - i$th coordinates by two different pairs of equal numbers. As we remarked above, closed orbits are parametrized by elements consisting of only signs. Since we have given the closure order and dimension formula above, the criterion of (5) becomes very explicit, and we may apply it directly as follows.

Fix a $\gamma = (c_1, \ldots, c_{2n}) \in \Sigma_\pm^{\text{sym}}(2p, 2q)$ including one of the bad patterns and not of the form treated by Lemma 3.1. We first produce a suitable closed orbit $\mathcal{O}_{cl}$ lying below $\mathcal{O}_\gamma$. Look first at the natural numbers occurring twice among $c_1, \ldots, c_n$. Replace the first occurrence of all such numbers by $+$ and the second by $-$. Then look at all the natural numbers occurring just once among $c_1, \ldots, c_n$. Whenever $c_i = c_{2n+1-j}$ is a natural number and $i < j < n$ (so that $c_j = c_{2n+1-i}$ is also a natural number), replace $c_i$ by $+$ and $c_j$ by $-$. Finally, replace $c_{n+1}, \ldots, c_{2n}$ by signs in such a way that the result $cl = (cl_1, \ldots, cl_{2n})$ is symmetric.

Assume for the moment that there are no indices $i < j \leqslant n$ with $c_i$ and $c_j$ equal natural numbers. Enumerate the indices $i \leqslant n$ with $c_i$ and $c_{2n+1-j}$ equal natural numbers for some $j \leqslant n$ as $i_1 < \cdots < i_{2m}$. Define a new element $\gamma' = (c'_1, \ldots, c'_{2n})$ by decreeing that $(c'_{i_1}, \ldots, c'_{i_1+2m-1}) = (1, \ldots, 2m)$, that $(c'_{2n+1-(2m-1)-i_1}, \ldots, c'_{2n+1-i_1}) = (2m-1, 2m, 2m-3, 2m-2, \ldots, 1, 2)$, and finally that there be as many $+$ signs among $(c'_1, \ldots, c'_n)$ as among $(c_1, \ldots, c_n)$ and similarly for $-$ signs. Then $\gamma'$ parametrizes an orbit $\mathcal{O}_{\gamma'}$ whose dimension is at least that of $\mathcal{O}_\gamma$, but for which the left-hand side of (5) for $\mathcal{O}_{\gamma'}$ (and $\mathcal{O}_{cl}$) is no larger than it is for $\mathcal{O}_\gamma$ (and $\mathcal{O}_{cl}$). Thus if either the closure of $\mathcal{O}_{\gamma'}$ is rationally singular by Springer's criterion, or this closure has dimension larger than that of the closure of $\mathcal{O}_\gamma$, then $\mathcal{O}_\gamma$ has rationally singular closure. Note that both sides of (5) are unaffected if we replace $\gamma'$ by the new (smaller) element obtained by deleting all the initial signs $c'_1, \ldots, c'_{i_1-1}$ and terminal signs $c'_{2n-i_1+2}, \ldots, c'_{2n}$ from $\gamma'$, and replace $cl$ by the element obtained by deleting the corresponding initial and terminal entries from $cl$. So we may assume $i_1 = 1$.

We argue inductively based on the number of distinct natural numbers $k$ appearing in $\gamma'$ as follows. We will reduce our analysis to the case of $k = 2$, so assume first that $k > 2$. Consider the new element $\gamma''$ (of smaller size) obtained from $\gamma'$ first by deleting the initial entries $c'_1, \ldots, c'_{i_2}$ and terminal entries $c'_{2n+1-i_2}, \ldots, c'_{2n}$, and then deleting all the initial and terminal signs from the result. Let $cl''$ denote the element obtained by deleting the corresponding entries from $cl$. Then in passing from $\mathcal{O}_{\gamma'}$ to $\mathcal{O}_{\gamma''}$ (and from $\mathcal{O}_{cl}$ to $\mathcal{O}_{cl''}$), the difference of the left- and right-hand sides of (5) will strictly increase in all cases unless $i_2 = 2$, in which case the difference will remain the same. Continuing this procedure, we are thus led to investigating the "base case" of the form $\gamma'' = (1, 2, \epsilon_3, \ldots, \epsilon_n, \epsilon_n, \ldots, \epsilon_3, 1, 2)$ and $cl'' = (+, -, \epsilon_3, \ldots, \epsilon_n, \epsilon_n, \ldots, \epsilon_3, -, +)$. There are two possibilities. If the signs $\epsilon_i$ are not all the same, one checks directly that (5) holds for the base case. Since each of the steps leading to the base case weakly increases the difference of the left- and right-hand sides of (5), one concludes that (5) holds for $\mathcal{O}_{\gamma'}$ and $\mathcal{O}_{cl}$, hence for $\mathcal{O}_\gamma$ and $\mathcal{O}_{cl}$, and hence the closure of $\mathcal{O}_\gamma$ is not rationally smooth, as claimed. The other possibility is that all of the signs in the base case are the same. In this case, the two sides of (5) corresponding to the base case are actually equal. Again since each step in the reduction weakly increases the difference of the left- and right-

hand sides of (5), the inequality in (5) holds for $\mathcal{O}'_\gamma$ and $\mathcal{O}_{cl}$ (and hence the closure of $\mathcal{O}_\gamma$ fails to be rationally smooth) except possibly in just one case, namely when each step reducing to the base case does not increase the difference of the left- and right-hand sides, and when the base case turns out to have all signs the same. But the only way this can happen is if $\gamma'$ is of the form $\gamma_\circ(p', q')$ (possibly flanked by a number of signs). If this is the case, the construction of $\gamma'$ shows that $\gamma$ must be of the same form up to a permutation of the natural number entries. Since we have assumed that $\gamma$ is not of the form treated by Lemma 3.1, one checks that the initial passage from $\gamma$ to $\gamma'$ indeed increases the differences between the left- and right-hand sides of (5). So once again the closure of $\mathcal{O}_\gamma$ is not rationally smooth, as desired.

Finally return to the case where there are indices $i < j \leqslant n$ with $c_i$ and $c_j$ equal natural numbers in $\gamma$, and replace $c_i, c_j$ for every such pair $i < j$ by signs as in the definition of $cl$ above (and define $c_{2n+1-i}, c_{2n+1-j}$ so that the resulting element is symmetric). We obtain an element parametrizing an orbit that either does not contain any of the bad patterns, or whose closure fails to be rationally smooth by the above argument. Changing the $i$th and $j$th coordinates of this element back to $c_i, c_j$ (and similarly for the $2n + 1 - i$th and $2n + 1 - j$th coordinates), we find that the dimension of the corresponding orbit increases, but so too does the left-hand side of (5) by at least the same number, and by a larger number if two of the signs between $c_i$ and $c_j$ are different. We conclude in all cases that the closure of $\mathcal{O}_\gamma$ is not rationally smooth, as desired. □

Finally we must consider the situation for the symmetric subgroup $\overline{K}$ of $\overline{G} = \mathrm{PSp}(2n, \mathbb{C})$ with Lie algebra $\mathfrak{sp}(2p, \mathbb{C}) \times \mathfrak{sp}(2q, \mathbb{C})$. Then $\overline{K}$ is connected (and its orbits on $\mathcal{B}$ coincide with those of $K$) *unless* $p = q$, in which case $\overline{K}$ has two components. In this case, orbits of $\overline{K}$ are parametrized by equivalence classes in $\Sigma_\pm^{\mathrm{sym}}(2p, 2q)$ generated for the relation generated by $\gamma \sim -\gamma$. Just as in the case discussed at the end of Section 2, if $\overline{\gamma}$ denotes such a class, the $\overline{K}$ orbit $\mathcal{O}_{\overline{\gamma}}$ parametrized by $\overline{\gamma}$ breaks into $K$ orbits as

$$\mathcal{O}_{\overline{\gamma}} = \mathcal{O}_\gamma \cup \mathcal{O}_{-\gamma}$$

which is disconnected if $\gamma \neq -\gamma$. Once again one may retrace the steps of the proof of Theorem 3.2 to show that no complications arise. One thus concludes: *Theorem* 3.2 *holds for the orbits of any symmetric subgroup $\overline{K}$ of $\overline{G}$ (isogenous to $\mathrm{Sp}(2p + 2q, \mathbb{C})$) with Lie algebra $\mathfrak{sp}(2p, \mathbb{C}) \oplus \mathfrak{sp}(2q, \mathbb{C})$.*

## 4. SO*($2n$)

We follow the same strategy as outlined at the beginning of Section 3. Let $\langle \cdot, \cdot \rangle$ denote a nondegenerate skew sesquilinear form on an $n$-dimensional quaternionic vector space $V$, so that

$$\langle u, v \rangle = -\overline{\langle v, u \rangle}.$$

As in the previous section, set

$$\langle \cdot, \cdot \rangle' = \mathrm{A} \circ \langle \cdot, \cdot \rangle$$

and

$$\langle \cdot, \cdot \rangle'' = \mathrm{B} \circ \langle \cdot, \cdot \rangle.$$

Then $\langle \cdot, \cdot \rangle'$ is a nondegenerate Hermitian form of signature $(n, n)$ on the underlying $2n$-dimensional complex vector space $V_\mathbb{C}$ and $\langle \cdot, \cdot \rangle''$ is a nondegenerate symmetric form on $V_\mathbb{C}$. Let $G_\mathbb{R} = \mathrm{SO}^*(2n)$ denote the subgroup of $\mathrm{GL}(V_\mathbb{H})$ preserving $\langle \cdot, \cdot \rangle$ and view $G_\mathbb{R}$ as subgroup of $\mathrm{GL}(V_\mathbb{C})$. Let $G'_\mathbb{R}$ denote the subgroup of $G' := \mathrm{GL}(V_\mathbb{C})$ preserving $\langle \cdot, \cdot \rangle'$, and let $G_\pm$ denote the subgroup of $\mathrm{GL}(V_\mathbb{C})$ preserving $\langle \cdot, \cdot \rangle''$. Then $G_\pm \simeq \mathrm{O}(2n, \mathbb{C})$, $G'_\mathbb{R} \simeq \mathrm{U}(n, n)$, and

$$G_\mathbb{R} = G'_\mathbb{R} \cap G_\pm.$$

(In fact, every element of $G_\mathbb{R}$ has determinant one, so $G_\mathbb{R}$ is indeed a real form of the connected algebraic group $G \simeq \mathrm{SO}(2n, \mathbb{C})$ consisting of determinant one elements in $G_+$.) Fixing compatible involutions $\theta$ and $\theta'$ as in Section 3, we naturally have

$$\mathrm{GL}(n, \mathbb{C}) \simeq K := G^\theta < K' := (G')^{\theta'} \simeq \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}).$$

Let $\mathcal{B}$ denote the flag variety for $G$ and $\mathcal{B}'$ the flag variety for $G'$. Once again, there is a natural inclusion $\mathcal{B}' \subset \mathcal{B}$, and if $\mathcal{O}'$ is an orbit of $K'$ on $\mathcal{B}'$, then its intersection with $\mathcal{B}$ is either empty or consists of a single orbit of $K$ on $\mathcal{B}$. If $\gamma' = (c'_1, \ldots, c'_{2n}) \in \Sigma_\pm(n, n)$, we say that it is antisymmetric if:

(i) If the entry $c'_i$ is a sign, then $c'_{2n+1-i}$ is the opposite sign.
(ii) If $c'_i = c'_j$ are natural numbers, then $j \neq 2n + 1 - i$ and $c'_{2n+1-i} = c'_{2n+1-j}$.
(iii) Among the entries $c'_1, \ldots, c'_n$, the number of $+$ signs plus the number of pairs of equal natural numbers is even.

Write $\Sigma_\pm^{\mathrm{asym}}(n, n)$ for the set of such antisymmetric elements. Notice there is a choice of sign in (iii) above. In Section 2, when $p = q = n$, the parametrization of orbits can obviously be twisted by an outer automorphism of $\mathrm{U}(n, n)$, the effect of which is to change all signs. Possibly after twisting the parametrization of Section 2, the $K'$ orbit $\mathcal{O}'_{\gamma'}$ meets $\mathcal{B}$ if and only if $\gamma'$ is antisymmetric. Thus $\Sigma_\pm^{\mathrm{asym}}(n, n)$ parametrizes $K$ orbits on $\mathcal{B}$ [MatOsh88]. (It may be helpful to refer to Figs. 3 and 4 in the course of the discussion below.)

Fix a $\theta'$-stable Cartan subalgebra $\mathfrak{h}'$, a choice of positive roots, and write $(\Delta')^+ = \{e'_i - e'_j \mid i < j\}$. Then $\mathfrak{h} := \mathfrak{g} \cap \mathfrak{h}'$ is a $\theta$-stable Cartan subalgebra of $\mathfrak{g}$. Choose a positive subset of roots of $\mathfrak{h}$ in $\mathfrak{g}$, in appropriate coordinates write $\Delta^+ = \{(e_i \pm e_j) \mid i < j\}$, and let $B$ denote the corresponding Borel subgroup of $G$. Fix a simple root $\alpha$ in $\Delta^+$ and let $S'(\alpha)$ the set of roots in $(\Delta')^+$ which restrict to $\alpha$. For instance, if $\alpha = e_i - e_{i+i}$, then $S'(\alpha) = \{\alpha', \alpha''\}$ with $\alpha' = e'_i - e'_{i+1}$ and $\alpha'' = e'_{2n-i} - e'_{2n-i+1}$.

Adopt notation analogous to that around Eq. (9), fix $\alpha = e_i - e_{i+1} \in \Delta^+$, and an antisymmetric element $\gamma$ of signature $(n, n)$. Then once again we have the identical conclusions of Eqs. (9)–(12). The situation for $\alpha = e_{n-1} + e_n$ is more subtle, however. Set $\alpha_{n-1} = e_{n-1} - e_n$, fix an antisymmetric element $\gamma$ of signature $(n, n)$, and let $s'$ denote the reflection in the simple root $e'_n - e'_{n+1}$ in $(\Delta')^+$. Recall the action described at the end of Section 2. Then

$$\dim(s_\alpha \cdot \mathcal{O}_\gamma) = 1 + \dim(\mathcal{O}_\gamma) \tag{14}$$

if and only if

$$\dim\big(s'_\beta \cdot \mathcal{O}'_{s' \times \gamma}\big) = 1 + \dim\big(\mathcal{O}'_{s' \times \gamma}\big) \tag{15}$$

for some (equivalently, any) root $\beta \in S'(\alpha_{n-1})$. Moreover, in this case, if $\mathcal{O}'_\delta$ is the dense $K'$ orbit in $s'_\beta \cdot \mathcal{O}'_\gamma$, then $\mathcal{O}'_{s' \times \delta}$ intersected with $\mathcal{B}$ is the dense $K$ orbit in $s_\alpha \cdot \mathcal{O}_\gamma$. Using this, one may deduce that the closure order of $K$ orbits on the level of $\Sigma_\pm^{\mathrm{asym}}(n, n)$ is once again the restriction of the order on $\Sigma_\pm(n, n)$ given in Section 2. The dimension of the orbit parametrized by a $\gamma \in \Sigma_\pm^{\mathrm{asym}}(n, n)$ is given by

$$\dim(\mathcal{O}_\gamma) = d(K) + (1/2)\big(l(\gamma) - \{t \in \mathbb{N} \mid c_s = c_t \in \mathbb{N} \text{ with } s \leqslant n < t \leqslant 2n + 1 - s\}\big)$$

where $d(K)$ is the dimension of the flag variety for $K$, namely $\frac{1}{2}n(n-1)$, and $l(\gamma)$ is as in (7). Closed orbits are once again parametrized by elements consisting only of signs, while the open orbit is parametrized by

$$\gamma_\circ(n, n) := (1, 2, \ldots, 2m - 1, 2m, 2m - 1, 2m, \ldots, 1, 2), \tag{16}$$

if $n = 2m$ is even and

$$\gamma_\circ(n, n) := (1, 2, \ldots, 2m, -, +, 2m - 1, 2m, \ldots, 1, 2), \qquad (17)$$

if $n = 2m + 1$ is odd. We let $\pm\gamma_\circ(n, n)$ denote either $\gamma_\circ(n.n)$ or the element obtained from inverting the signs in $\gamma_\circ(n, n)$ (which differs from $\gamma_\circ(n.n)$ only if $n$ is odd and, in this case, no longer satisfies condition (iii) above).

We record two more operations on $\Sigma_\pm^{\mathrm{asym}}(n, n)$. Fix $\gamma \in \Sigma_\pm^{\mathrm{asym}}(n, n)$ and let $s'$ be as above. Let $\gamma'$ denote the element obtained by changing all signs in $s' \times \gamma$ (described at the end of Section 2). Then exactly one element of $\{\gamma', s' \times \gamma'\}$ is antisymmetric. Let $\tau(\gamma)$ denote this element. Thus $\tau$ is an involution on $\Sigma_\pm^{\mathrm{asym}}(n, n)$, and hence can be interpreted as an involution of the set of $K$ orbits on $\mathcal{B}$. It coincides with the action of an outer automorphism of $G$. (As an example, $\tau$ corresponds to the obvious symmetry in Figs. 3 and 4 below.) Finally, if $\gamma \in \Sigma_\pm(r, s)$, denote by $\gamma^{-r}$ the element obtained from $\gamma$ by reversing its coordinates, changing all of its signs, and replacing every pair of equal natural numbers by a different pair of equal natural numbers. Then the concatenation $(\gamma, \gamma^{-r})$ is antisymmetric and so may be viewed as an element of $\Sigma_\pm^{\mathrm{asym}}(r + s, r + s)$.

**Lemma 4.1.** *Fix* $\gamma \in \Sigma_\pm^{\mathrm{asym}}(n, n)$.

(a) *Suppose that there are integers* $r + s + n' = n$ *so that* $\gamma$ *can be written as the concatenation*

$$\big(\gamma_1, \pm\gamma_\circ(n', n'), \gamma_1^{-r}\big)$$

*where* $\pm\gamma_\circ(n', n')$ *is defined after* (17), $\gamma_1 \in \Sigma_\pm(r, s)$ *which avoids the bad patterns of Theorem 2.2, and* $\gamma_1^{-r}$ *is defined as above. Then the closure of* $\mathcal{O}_\gamma$ *is a fiber bundle with smooth fiber over a partial flag variety for* $K$, *and hence is smooth. The fiber is isomorphic to the product of the flag variety for* $\mathrm{SO}(2n', \mathbb{C})$ *and the closure of the* $\mathrm{GL}(r, \mathbb{C}) \times \mathrm{GL}(s, \mathbb{C})$ *orbit parametrized by* $\gamma_1$ *(as in Section* 2).

(b) *Suppose* $n = 2m$ *is even and that* $\gamma$ *can be written as*

$$\big(1, \gamma_1, 2, 1, \gamma_1^{-r}, 2\big),$$

*where* $\gamma_1 \in \Sigma_\pm(r - 1.s - 1)$, $r + s = m$, *avoids the bad patterns of Theorem 2.2. Then the closure of* $\mathcal{O}_\gamma$ *is a fiber bundle with smooth fiber over a partial flag variety for* $K$, *and hence is smooth. The fiber is isomorphic to the closure of the* $\mathrm{GL}(r, \mathbb{C}) \times \mathrm{GL}(s, \mathbb{C})$ *orbit parametrized by the element* $(1, \gamma_1, 1) \in \Sigma_\pm(r, s)$.

**Proof.** Part (a) is proved much the same way as Lemma 3.1 and we omit the details. For (b), note that the closure of the orbit parametrized by $\gamma$ is isomorphic to the one parametrized by the outer automorphism conjugate $\tau(\gamma)$ described above. But if $\gamma$ has the form indicated in (b), then $\tau(\gamma)$ has the form indicated in (a). Thus (b) follows from (a). □

**Theorem 4.2.** *Fix* $\gamma \in \Sigma_\pm^{\mathrm{asym}}(n, n)$, *an antisymmetric involution of* $S_{2n}$ *with signed fixed points of signature* $(n, n)$ *(as defined above). Suppose* $\gamma$ *is not of the form treated by Lemma* 4.1 *and, further, that* $\gamma$ *includes one of the patterns* $(1, +, -, 1)$, $(1, -, +, 1)$, *and* $(1, 2, 1, 2)$. *Then the closure of* $\mathcal{O}_\gamma$ *does not have rationally smooth closure.*

*In all other cases, the closure of* $\mathcal{O}_\gamma$ *is a fiber bundle with smooth fiber over a partial flag variety for* $K$, *and hence is smooth. In this case, there are integers* $r + s + n' = n$ *so that the fiber is isomorphic to the product of the flag variety for* $\mathrm{SO}(2n', \mathbb{C})$ *with the closure of an orbit of* $\mathrm{GL}(r) \times \mathrm{GL}(s, \mathbb{C})$ *on the flag variety for* $\mathrm{GL}(r + s, \mathbb{C})$ *with* $r + s = n$.

*In particular,* $\mathcal{O}_\gamma$ *has rationally smooth closure if and only if it has smooth closure, or if and only if the condition of Theorem* 1.1 *fails for some closed orbit below it.*

**Proof.** This is very similar to the proof of Theorem 3.2. We omit the details. □

Finally we turn to issues of isogeny. First assume $n$ is odd, so that the center of $\mathrm{Spin}(2n, \mathbb{C})$ is $\mathbb{Z}/4$. So the three complex groups $\overline{G}$ to consider are $\mathrm{Spin}(2n, \mathbb{C})$, $\mathrm{SO}(2n, \mathbb{C})$, and $\mathrm{PSpin}(2n, \mathbb{C})$. It turns out that the symmetric subgroup $\overline{K}$ with Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ is connected in each of these cases, and that the orbits of $\overline{K}$ on $\mathcal{B}$ are insensitive to isogeny. So Theorem 4.2 applies without change.

The case of $n = 2m$ even is more interesting since the center of $\mathrm{Spin}(2n, \mathbb{C})$ is $\mathbb{Z}/2 \times \mathbb{Z}/2$. Write $\mathrm{SO}(2n, \mathbb{C})$ for the quotient by the diagonal $\mathbb{Z}/2$, $\mathrm{SO}'(2n, \mathbb{C}) \simeq \mathrm{SO}''(2n, \mathbb{C})$ for either of quotients by an off-diagonal $\mathbb{Z}/2$, and $\mathrm{PSpin}(2n, \mathbb{C})$ for the adjoint group. Write $K_{\mathrm{sc}}$, $K$, $K'$, and $K_{\mathrm{ad}}$ for the corresponding symmetric subgroups with Lie algebra $\mathfrak{gl}(n, \mathbb{C})$. The orbits of $K_{\mathrm{sc}}$ and $K$ on $\mathcal{B}$ always coincide, so are treated by Theorem 4.2. If $m$ is even, these orbits also coincide with the orbits of $K'$ on $\mathcal{B}$. If $m$ is odd, the orbits of $K'$ instead coincide with the orbits of $K_{\mathrm{ad}}$ on $\mathcal{B}$. Finally, the orbits of $K_{\mathrm{ad}}$ are parametrized by equivalence classes in $\Sigma_{\pm}^{\mathrm{asym}}(n, n)$ for the relation generated by $\gamma \sim \tau(\gamma)$ for the involution of $\Sigma_{\pm}^{\mathrm{asym}}(n, n)$ described above. If $\overline{\gamma}$ denotes such a class, the $\overline{K}$ orbit $\mathcal{O}_{\overline{\gamma}}$ parametrized by $\overline{\gamma}$ breaks into $K$ orbits as

$$\mathcal{O}_{\overline{\gamma}} = \mathcal{O}_{\gamma} \cup \mathcal{O}_{\tau(\gamma)}$$

which is disconnected if $\gamma \neq \tau(\gamma)$. Nonetheless the (omitted) proof of Theorem 4.2 goes through unchanged to establish that the statement of the theorem requires no modification in this case.

The conclusion of the discussion is: *Theorem 4.2 holds for the orbits of any symmetric subgroup $\overline{K}$ of $\overline{G}$ (isogenous to $\mathrm{Spin}(2n, \mathbb{C})$) with Lie algebra $\mathfrak{gl}(n, \mathbb{C})$.*

## 5. Examples

We conclude with several examples of the closure order of $K$ orbits on $\mathcal{B}$. Vertices in the diagrams below correspond to orbits, and orbits of the same dimension appear on the same column (or, in the case of Fig. 2, row). Recall that the closure order is generated by relations in codimension one; so each diagram need only keep track of such relations. Dashed edges correspond to relations not present in the weak closure order, as described at the beginning of Section 2.

Fig. 1 corresponds to the case of $\mathrm{SU}(2, 2)$; that is, the case where $G = \mathrm{SL}(4, \mathbb{C})$, $K = \mathrm{S}(\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C}))$, and $\mathcal{B}$ consists of complete flags in $\mathbb{C}^4$. (Without the dashed edges and boxed vertices this is [MatOsh88, Fig. 7].) The labeling of simple roots is given by
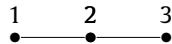


The boxed vertices correspond to orbits with nonsmooth closures according to the pattern criterion given in Theorem 2.2. Suppose now $\overline{G}$ is a nontrivial quotient of $G$ by a central subgroup, and let $\overline{K}$ be the corresponding symmetric subgroup of $\overline{G}$. According to the discussion at the end of Section 2, quotienting Fig. 1 by the obvious $\mathbb{Z}/2$ symmetry gives the corresponding picture for $\overline{K}$ orbits on $\mathcal{B}$.

Consider next the case of $\mathrm{Sp}(2, 2)$ where $G = \mathrm{Sp}(8, \mathbb{C})$ and $K = \mathrm{Sp}(4, \mathbb{C}) \times \mathrm{Sp}(4, \mathbb{C})$. According to the details of Section 3, there are 42 orbits for $\mathrm{Sp}(2, 2)$, too many to fit in a reasonable diagram. Instead consider $\overline{G} = \mathrm{PSp}(8, \mathbb{C}) = G/F$ where $F$ is the two element center of $G$. Then $\overline{K} = K/F$ is a symmetric subgroup of $\overline{G}$ corresponding to $\mathrm{PSp}(2, 2)$. The orbits of $\overline{K}$ on the flag variety are given in Fig. 2. (This graph, without the dashed edges and boxed vertices, is [MatOsh88, Fig. 15].) Simple roots are labeled as



According to the discussion at the end of Section 3, an orbit labeled by an element of $\Sigma_{\pm}^{\mathrm{sym}}(2, 2)$ which contain signs is the (disconnected) union of two $K$ orbits. (Nonetheless, such an orbit has
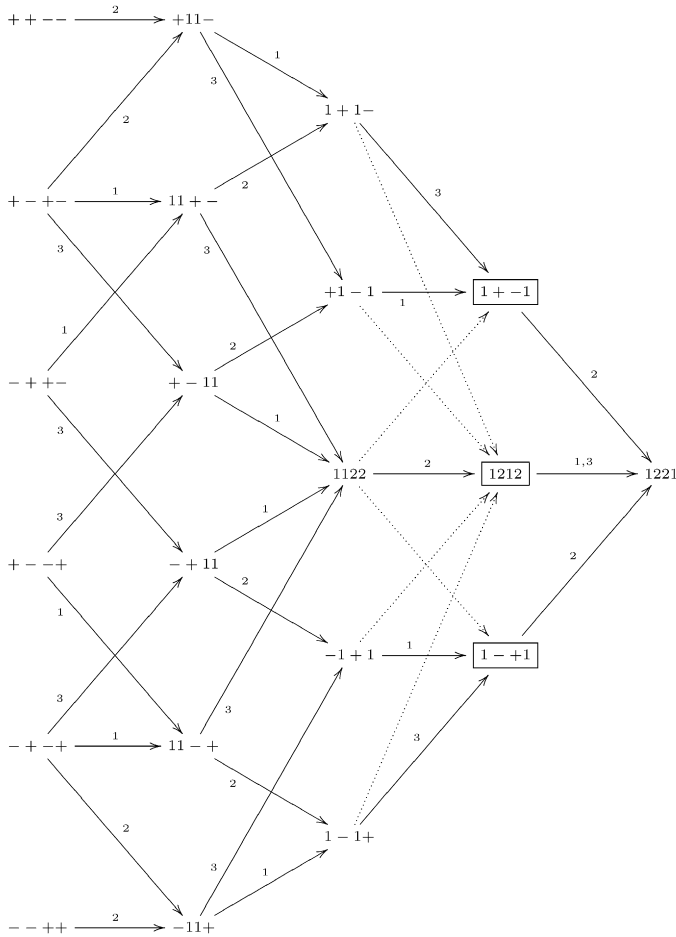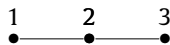
**Fig. 1.** U(2, 2).

smooth closure if and only if each connected component has smooth closure.) Boxed vertices in Fig. 2 correspond to $\overline{K}$ orbits with nonsmooth closures according to the criterion of Theorem 3.2.

Fig. 3 gives the case of SO*(6), i.e. $K = \mathrm{GL}(3, \mathbb{C})$ orbits acting on the flag variety for $\mathfrak{so}(6, \mathbb{C}) \simeq \mathfrak{sl}(4, \mathbb{C})$. (This diagram is [MatOsh88, Fig. 20].) Simple roots are labeled as



Since SO*(6) is a quotient of SU(3, 1) by the subgroup $\mathbb{Z}/2$ (defined over $\mathbb{R}$) of the center $\mathbb{Z}/4$ of Spin(6, $\mathbb{C}$) = SL(4, $\mathbb{C}$), either Theorem 2.2 (and the discussion at the end of Section 2) or Theorem 4.2 applies to give that all orbits have smooth closure in this case.

Finally, Fig. 4 gives the case of SO*(8), i.e. GL(4, $\mathbb{C}$) orbits on the flag variety for $\mathfrak{so}(8, \mathbb{C})$. (This diagram without the dashed edges or boxed vertices is [MatOsh88, Fig. 19].) Simple roots are labeled as
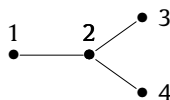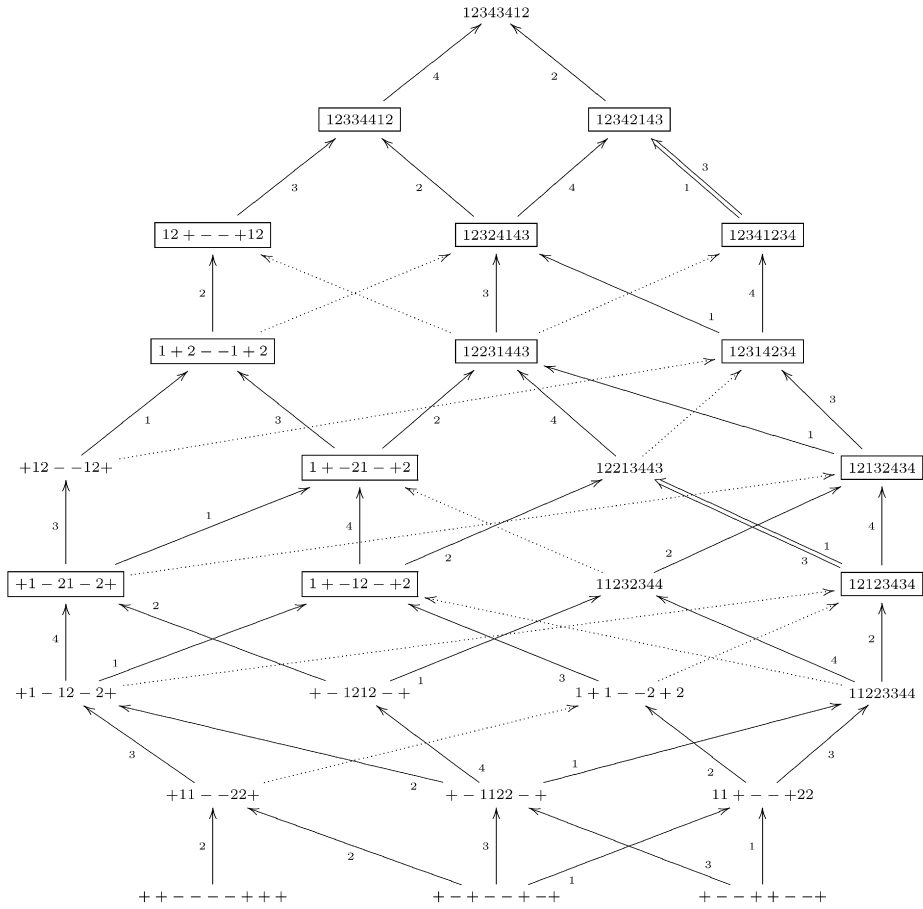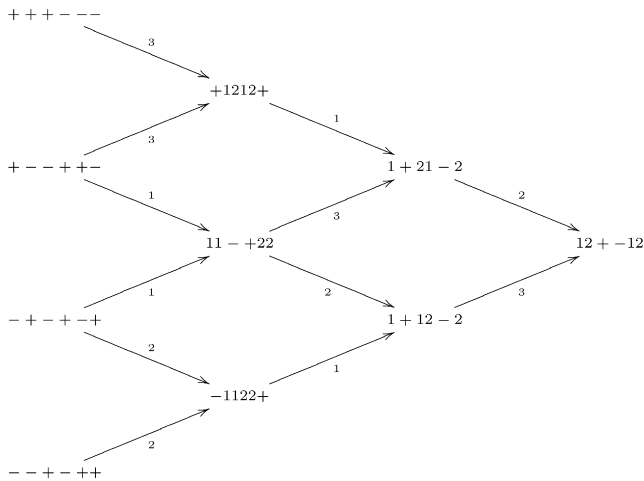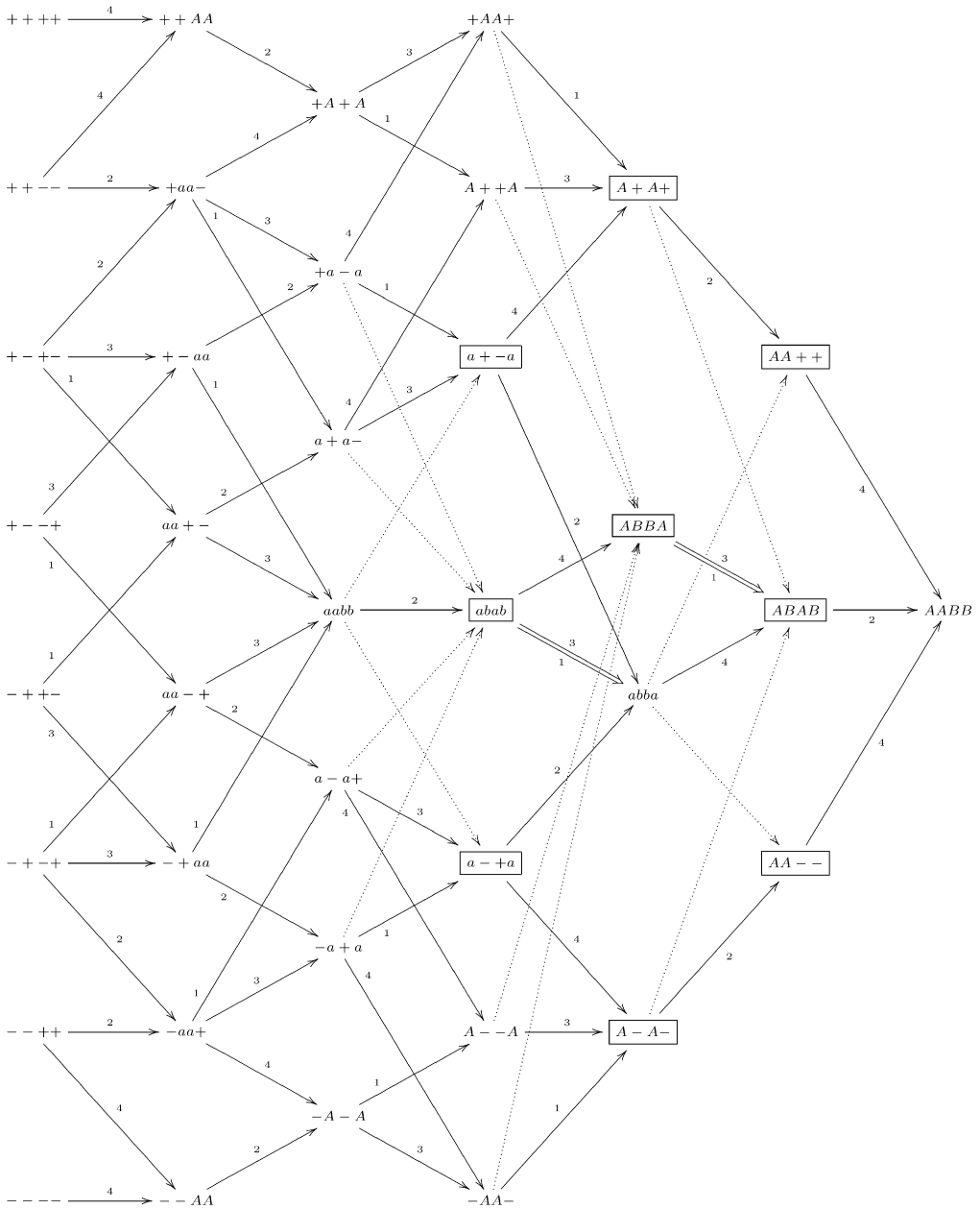
**Fig. 2.** PSp(2, 2).



**Fig. 3.** SO*(6).

**Fig. 4.** SO*(8).

To conserve space, we introduce the following shorthand (as in [MatOsh88]). The eight symbols of an element $\gamma \in \Sigma_{\pm}^{\mathrm{asym}}(4, 4)$ are compressed to just four. The signs in the compressed symbol match those in the first four coordinates of $\gamma$; a pair of numbers in $\gamma$ in positions $i$ and $j$ with $1 \leqslant i < j \leqslant 4$ is represented by a lower-case letter in positions $i$ and $j$ of the compressed symbol; and a pair of numbers in $\gamma$ in positions $i \leqslant 4$ and $9 - j > 4$, is represented by an upper-case letter in positions $i$ and $j$ of the compressed symbol. So, for instance, the $\gamma = 1 + - 12 + - 1$ becomes $a + - a$;

$+12 + -12-$ becomes $+AA+$; $12123434$ becomes $abab$; and $12341234$ becomes $ABBA$. With this convention, the boxed vertices correspond to orbits with nonsmooth closures according to the condition of Theorem 4.2. Isogeny considerations amount (possibly) to folding the figure by the obvious symmetry, as explained at the end of Section 4.

**Example 5.1.** We conclude with some examples where isogeny considerations necessarily make formulating pattern avoidance results more complicated and less uniform. Let $G = \text{Sp}(2n, \mathbb{C})$, $\overline{G} = \text{PSp}(2n, \mathbb{C}) = G/F$, $K = \text{GL}(n, \mathbb{C})$, and $\overline{K} = K/F$. By considerations similar to those treated in Section 3, orbits of $K$ on $\mathcal{B}$ are parametrized by $\Sigma_{\pm}^{\text{asym}}(2n)$, the union over all $p + q = n$ of antisymmetric elements in $\Sigma_{\pm}(p, q)$. Orbits of $\overline{K}$ are parametrized by equivalence classes in $\Sigma_{\pm}^{\text{asym}}(2n)$ for the relation generated by $\gamma \sim -\gamma$ and the obvious version of (8) holds. This time, however, the relationship between the (rational) smoothness of the closure of $\mathcal{O}_{\overline{\gamma}}$ and the (rational) smoothness of the closures of its connected components is a little complicated. For instance, let $\gamma_2 = 1 + -1$, $\gamma_3 = +1 + -1-$, and $\gamma_4 = 112 + -233$. Then each $\gamma_i$ contains (in the sense of Definition 2.1) the pattern $1 + -1$. The $K$ orbits $\mathcal{O}_{\gamma_i}$ each have closures which are smooth but not rationally smooth. Meanwhile the $\overline{K}$ orbit $\mathcal{O}_{\overline{\gamma}_1}$ has closure which is not rationally smooth, $\mathcal{O}_{\overline{\gamma}_2}$ has closure which is rationally smooth (but not smooth), and $\mathcal{O}_{\overline{\gamma}_3}$ has closure which once again is not rationally smooth. (One may prove the rational smoothness assertions using Theorem 1.1; so, in particular, the criterion of the theorem is sensitive to isogeny.) Further calculation suggest a relatively simple pattern avoidance criterion for (rational) smoothness of $K$ orbit closures may exist, but formulating such a result for $\overline{K}$ orbit closures is messier. The situation is similarly complicated for $\text{SO}(p, q)$ (and even more so if $p + q$ is divisible by four). This example suggests that it is perhaps reasonable to assume that $G$ is simply connected (and thus $K$ is connected) when formulating pattern avoidance results in general.

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