SOME SMALL UNIPOTENT REPRESENTATIONS OF INDEFINITE ORTHOGONAL GROUPS AND THE THETA CORRESPONDENCE

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Abstract. We locate a family of small unitary representations of the orthogonal groups in the theta correspondence for the dual pairs $\left(\mathrm{Sp}(2n,\mathbb{R}),\mathrm{O}(p,q)\right)$, identifying them as double stable range lifts of the trivial representation of $\mathrm{O}(n)$. We simultaneously study the corresponding lifts of the determinant character, and show that the two lifts are irreducible constituents of unipotent derived functor modules on the "edge" of the weakly fair range.

1. INTRODUCTION

Consider a sequence of reductive dual pairs $(G_1,G_2),(G_2,G_3),(G_3,G_4),\ldots,(G_k,G_{k+1})$. An old idea is that under suitable hypotheses one should obtain interesting small unitary representations of G_k by beginning with interesting small unitary representations of G_1 and performing a sequence of iterated theta lifts. In particular, one could begin with the simplest unitary representations of G_1, the one-dimensional ones. This framework of iterated theta-lifting then provides a way to organize unitary representations of classical groups. It is natural to ask to what extent theta lifting preserves finer invariants of unitary representations. Questions of this sort have been studied by many authors. The best results of this kind are due to Howe and Li ([19]) and characterize low-rank unitary representations; see also Przebinda's paper [26] for other successes. Our interest here is somewhat more qualitative: suppose that π and π' are two closely related unitary representations of G_1 — for instance, suppose π and π' differ by tensoring with a character of G_1 — then how are their iterated lifts related? This is often very difficult to make explicit. The purpose of this paper is to establish some results in this direction. The virtue of their formulation is that they immediately suggest generalizations.

We work with the following sequence of pairs: $\left(\mathrm{O}(s,0),\mathrm{Sp}(2m,\mathbb{R})\right),\left(\mathrm{Sp}(2m,\mathbb{R}),\mathrm{O}(2m,r)\right)$ with $s\le m\le r/2$; these latter inequalities correspond to the stable range. Given a representation π of $\mathrm{O}(s,0)$, we let $\theta^2(\pi)$ denote the corresponding double lift to $\mathrm{O}(2m,r)$. (Because of the covers involved, this notation is imprecise but adequate for the introduction; more complete details are given in Section 2.2.) Let $\mathbf{1}_s$ and $\det_s$ denote the trivial and determinant representations of $\mathrm{O}(s,0)$. Our first result (Theorem 1.2) identifies the double lift $\theta^2(\mathbf{1}_s)$ as the special unipotent representation π_s' introduced by Knapp in [13] and [14] and studied further in [29]. We remark that a study of all double lifts of compact groups has recently been completed by Loke and Nishiyama-Zhu. We return to this below.

Before stating the theorem, we recall the definition of the representations of [14]. Let G be the identity component $\mathrm{SO}_e(2m,r)$ and assume $m\le r/2$. Write $\mathfrak{g}$ for the complexified Lie algebra of G and τ for the complexified Cartan involution. Let $l=\left[\frac{2m+r}{2}\right]$, the rank of $\mathfrak{g}$. Fix an integer $s\ge 0$ whose parity matches that of r. (This latter condition may be dropped if we pass to the nonlinear cover of G, but since those groups do not arise in the theta correspondence we impose the parity condition.) Consider a τ-stable parabolic subalgebra $\mathfrak{q}=\mathfrak{l}\oplus\mathfrak{u}$ in $\mathfrak{g}$ whose Levi factor corresponds to the subgroup

$$L=\mathrm{U}(m,0)\times\mathrm{SO}_e(0,r)\subset G.$$

Both authors were partially supported by NSF grant DMS-0532088. The second author was also partially supported by NSF grant DMS-0300106.

Let $\mathbb{C}_{\lambda_s}$ be the one-dimensional representation of L which is the $(-l+\frac{s}{2})$th power of the determinant representation on the $\mathrm{U}(m,0)$ factor of L and trivial on the SO factor, and consider the derived functor module

$$\pi_s = A_{\mathfrak{q}}(\lambda_s). \tag{1.1}$$

Our notation follows that of [15]; in particular there is a ρ-shift involved in the infinitesimal character. The definition is arranged so that π_s is in the weakly fair range whenever $s \geq m+1$, and it turns out (see [29]) that indeed π_s is irreducible whenever $s \geq m$. As soon as $s < m$, π_s becomes reducible, however, and we let π'_s denote the unique irreducible constituent of π_s whose lowest K-type matches the lowest K-type of π_s; see Sections 2.6 and 2.7 for more precise details. (This construction is an algebraic version of the kinds of analytic continuations first considered by Wallach in [36] and later taken up by other authors, e.g. [5], [8].) Knapp proves that whenever $0 \leq s_1 < s_2 \leq m$, π'_{s_1} is a unitary representation whose Gelfand-Kirillov dimension is strictly less than that of π'_{s_2}. Thus for $s > m$, the representation π'_s is a unitary cohomologically induced representation of the form $A_{\mathfrak{q}}(\lambda)$. But for $s < m$, π'_s is an interesting small unitary representation that is not obviously cohomologically induced. These latter representations arise very naturally in the theta correspondence. (A minor complication is that π'_s is defined as a representation of the identity component of $\mathrm{O}(2m,r)$ while it is representations of the full orthogonal group that arise in the correspondence.)

Theorem 1.2. *Fix integers $s \leq m \leq r/2$ so that the parity of r matches that of s. When restricted to the identity component $\mathrm{SO}_e(2m,r)$, the iterated lift of the trivial representation of $\mathrm{O}(s,0)$ to $\mathrm{Sp}(2m,\mathbb{R})$ to $\mathrm{O}(2m,r)$ contains the representation π'_s as a summand.*

Because of some innocuous choices involved in defining the theta correspondence, Theorem 1.2 (and Proposition 1.3 below) are stated slightly imprecisely. See Section 2.2 and the statement of Theorem 2.1 (and Proposition 2.2).

As we mentioned above, Loke and Nishiyama-Zhu have also recently studied double lifts of representations of a compact group [20], [22], [23]. For instance, they give explicit formulas for the restriction of such double lifts to a maximal compact subgroup, say K. Since such formulas are also available for Knapp's representations [29], Theorem 1.2 may be proved by combining [20], [23], and [29]. But our interest here is somewhat different: we seek to identify certain double lifts as special unipotent representations and interpret them in terms of cohomological induction. The virtue of this formulation is that our results suggest generalization beyond double lifts from compact groups. Since it is difficult to recognize singular derived functor modules from their K-spectrums, we develop an alternative route to Theorem 1.2 based on the uniqueness statement given in Proposition 4.1. That approach makes the relationship between the double lifts $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ more transparent. In particular, it is closely connected to determining their Langlands parameters (which we do in Section 5).

We now discuss in more detail how the double lift $\theta^2(\det_s)$ is related to $\theta^2(\mathbf{1}_s)$. As we explain in Section 4, this matter is eventually reduced (in the notation of Theorem 1.2) to the case of $s = m$ and $r = 2m$ (or $r = 2m+1$). In this case, $\pi'_m = \pi_m$ and $\pi'_{m+2} = \pi_{m+2}$; that is, the full cohomologically induced representations π_m and π_{m+2} are both irreducible. In fact, π_m and π_{m+2} "straddle" the edge of the weakly fair range in the sense that π_k for $k \geq m+1$ is in the weakly fair range, but π_m is not. Moreover, π_m and π_{m+2} are even more closely related in that they have the same infinitesimal character, annihilator, and associated variety. (They are both special unipotent representations attached to the same dual nilpotent orbit.) We have:

Proposition 1.3. *Fix an integer $m > 0$ and let $r = 2m$ if m is even and $r = 2m+1$ if m is odd. When restricted to $\mathrm{SO}_e(2m,r)$, the iterated lift of the determinant representation of $\mathrm{O}(m,0)$ to $\mathrm{Sp}(2m,\mathbb{R})$ to $\mathrm{O}(2m,r)$ contains the representation π'_{m+2}. (Thus the representations $\theta^2(\mathbf{1}_m)$ and $\theta^2(\det_m)$ straddle the edge of the weakly fair range in the sense described above.)*

As mentioned above, a general (but slightly more technical) statement analogous to Proposition 1.3 for $s \neq m$ is given in Proposition 4.6. In more detail, we understand the relationship between $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ by inducing them up to well-understood special unipotent representations of a larger group. (The technique of understanding unipotent representations of a smaller group in terms of those of a larger one is an old idea; for instance, it is one of the main tools in the inductive description of the results of [3].) In the end, loosely speaking, one may say that for all s, $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ again straddle the edge of the weakly fair range.

Acknowledgments. We thank Tony Knapp for several helpful correspondences.

2. EXPLICIT DETAILS CONCERNING π'_s

In this section, we introduce some auxiliary notation and recall some properties of the representation π'_s defined in the introduction.

2.1. **General notation.** Throughout G will denote a reductive Lie group with Lie algebra $\mathfrak{g}_\circ$ and complexification $\mathfrak{g}$. We let $G_\mathbb{C}$ denote the connected complex adjoint group associated to $\mathfrak{g}$. Recall that $G_\mathbb{C}$ acts on the nilpotent cone in $\mathfrak{g}$ with finitely many orbits; we let $\mathcal{N}$ denote the set of these orbits. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let W denote the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$. We let $\mathfrak{g}^\vee$ denote the complex dual Lie algebra; with $\mathfrak{h}$ fixed, $\mathfrak{g}^\vee$ comes equipped with a Cartan subalgebra $\mathfrak{h}^\vee$ which is canonically isomorphic to $\mathfrak{h}^*$, the linear dual of $\mathfrak{h}$. We let $\mathcal{N}^\vee$ denote orbits of $G^\vee_\mathbb{C}$ (the connected complex adjoint group defined by $\mathfrak{g}^\vee$) on the nilpotent cone in $\mathfrak{g}^\vee$.

Let K denote the maximal compact subgroup of G and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the complexified Cartan decomposition. Write $K_\mathbb{C}$ for the complexification of K. Then $K_\mathbb{C}$ acts with finitely many orbits on the set of nilpotent elements in $\mathfrak{p}$. We denote this set of orbits by $\mathcal{N}_\mathfrak{p}$.

2.2. **Notation for the theta correspondence.** Suppose (G, G') is a reductive dual pair in $\mathrm{Sp}(2n,\mathbb{R})$. Let $\mathrm{Mp}(2n,\mathbb{R})$ denote the connected double cover of $\mathrm{Sp}(2n,\mathbb{R})$. Let $\widetilde{G}$ and $\widetilde{G}'$ denote the preimages of G and G' in $\mathrm{Mp}(2n,\mathbb{R})$. Let $\mathrm{Irr}_{\mathrm{gen}}(\widetilde{G})$ denote the set of equivalence classes of irreducible Harish-Chandra modules for $\widetilde{G}$ that do not factor to G and adopt similar notation for $\mathrm{Irr}_{\mathrm{gen}}(\widetilde{G}')$. The theta correspondence is a map

$$\theta \;:\; \mathrm{Irr}_{\mathrm{gen}}(\widetilde{G}) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{G}') \cup \{0\};$$

here if $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\widetilde{G})$ does not occur in the correspondence we write $\theta(\pi) = 0$. The map θ depends on a choice of oscillator for $\mathrm{Mp}(2n,\mathbb{R})$.

It is often desirable to work directly with Harish-Chandra modules for G rather than genuine representations of $\widetilde{G}$. This is possible only if the cover $\widetilde{G}$ splits. In that case, there exist genuine characters of $\widetilde{G}$. Fix one such η. Then tensoring with η provides a bijection between $\mathrm{Irr}_{\mathrm{gen}}(\widetilde{G})$ and $\mathrm{Irr}(G)$.

We introduce some further notation in one special case. For $s \le m \le r/2$, consider the dual pairs $(\mathrm{O}(s,0), \mathrm{Sp}(2m,\mathbb{R}))$ and $(\mathrm{Sp}(2m,\mathbb{R}), \mathrm{O}(2m,r))$. The covers $\widetilde{\mathrm{O}}(s,0)$ and $\widetilde{\mathrm{O}}(2m,r)$ both split, so fix genuine characters η_1 and η_2 of them. Write

$$\theta_1 \;:\; \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{O}}(s,0)) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{Sp}}(2m,\mathbb{R})),$$

and

$$\theta_2 \;:\; \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{Sp}}(2m,\mathbb{R})) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{O}}(2m,r)).$$

(The conditions that $s \le m \le r/2$, i.e. that each pair is in the stable range, dictates that all lifts are nonzero.) Consider the following composition

$$\mathrm{Irr}(\mathrm{O}(s,0)) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{O}}(s,0)) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{Sp}}(2m,\mathbb{R})) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{O}}(2m,r)) \longrightarrow \mathrm{Irr}(\mathrm{O}(2m,r))$$

defined by

$$X \longrightarrow X \otimes \eta_1 \longrightarrow \theta_1(X \otimes \eta_1) \longrightarrow \theta_2\left[\theta_1(X \otimes \eta_1)\right] \longrightarrow \theta_2\left[\theta_1(X \otimes \eta_1)\right] \otimes \eta_2.$$

We denote this composition by θ^2. It depends on choices of η_1, η_2, and of the oscillators defining θ_1 and θ_2. We make the standard choices, i. e., those used, e. g., in [21], [1], [24], and the references found in these papers. For η_1 this means that it is the unique character of $\widetilde{\mathrm{O}}(s,0)$ which occurs in the correspondence for every dual pair $\big(\mathrm{O}(s,0),\mathrm{Sp}(2k,\mathbb{R})\big)$ (for which $\mathrm{O}(s,0)$ has the same cover $\widetilde{\mathrm{O}}(s,0)$). The possible choices for η_2 differ by a character of $\mathrm{O}(2m,r)$; this choice does not affect the restriction of a representation to the identity component $\mathrm{SO}_e(2m,r)$.

With these choices in mind, we now restate Theorem 1.2 and Proposition 1.3 from the introduction.

Theorem 2.1. *Retain the notation introduced above. Fix integers $s \le m \le r/2$ so that the parity of r matches that of s. Then $\theta^2(\mathbf{1}_s)$ restricted to $\mathrm{SO}_e(2m,r)$ contains π'_s as a summand.*

Proposition 2.2. *Retain the notation introduced above. Fix an integer $m > 0$ and let $r = 2m$ if m is even and $r = 2m+1$ if m is odd. Then $\theta^2(\det_m)$ restricted to $\mathrm{SO}_e(2m,r)$ contains π'_{m+2} as a summand.*

2.3. **Primitive ideals: generalities.** A two-sided ideal in the enveloping algebra $\mathrm{U}(\mathfrak{g})$ is called primitive if it is the annihilator of a simple $\mathrm{U}(\mathfrak{g})$ module. Since each such simple module has an infinitesimal character, i.e. is annihilated by a unique codimension-one ideal in the center $\mathrm{Z}(\mathfrak{g})$, it is clear that each primitive ideal also contains a unique codimension-one ideal in $\mathrm{Z}(\mathfrak{g})$. Such ideals are parametrized (via the Harish-Chandra isomorphism) by W orbits on $\mathfrak{h}^*$. Let $\mathrm{Prim}(\mathfrak{g})$ denote the set of primitive ideals in $\mathrm{U}(\mathfrak{g})$ and $\mathrm{Prim}_\chi(\mathfrak{g})$ those with infinitesimal character $\chi \in \mathfrak{h}^*/W$. Duflo proved that $\mathrm{Prim}_\chi(\mathfrak{g})$ is finite and (in the inclusion partial order) contains a unique maximal element $\mathrm{J}_{\max}(\chi)$.

The associated variety $\mathrm{AV}(I)$ of $I \in \mathrm{Prim}(\mathfrak{g})$ is defined as follows. The ideal I inherits a grading from the obvious grading on $\mathrm{U}(\mathfrak{g})$. The associated graded $\mathrm{gr}(I)$ is a two-sided ideal in $\mathrm{gr}(\mathrm{U}(\mathfrak{g})) = S(\mathfrak{g})$. Thus $\mathrm{gr}(I)$ cuts out a subvariety, denoted $\mathrm{AV}(I)$, of $\mathfrak{g}^* \simeq \mathfrak{g}$. According to a well-known result of Borho-Brylinski, $\mathrm{AV}(I)$ is the closure of a unique element of $\mathcal{N}$.

2.4. **Special unipotent representations.** Given an orbit $\mathcal{O}^\vee$ in $\mathcal{N}^\vee$, we may construct a Jacobson-Morozov triple $\{e^\vee, h^\vee, f^\vee\}$ with $e^\vee \in \mathcal{O}^\vee$ and $h^\vee \in \mathfrak{h}^\vee$. We define $\chi(\mathcal{O}^\vee) = \frac{1}{2}h^\vee \in \mathfrak{h}^\vee \simeq \mathfrak{h}$. Different choices in this construction lead to at most a Weyl group translate of $\chi(\mathcal{O}^\vee)$. Hence $\chi(\mathcal{O}^\vee)$ is a well-defined element of $\mathfrak{h}^*/W$, and thus defines an infinitesimal character. We write $\mathrm{J}_{\max}(\mathcal{O}^\vee)$ for the maximal ideal $\mathrm{J}_{\max}(\chi(\mathcal{O}^\vee))$. Recall the Spaltenstein duality map

$$d\ : \mathcal{N} \longrightarrow \mathcal{N}^\vee,$$

as treated in the appendix to [2]. According to [2, Corollary A3],

$$\mathrm{AV}(\mathrm{J}_{\max}(\mathcal{O}^\vee) = \overline{d(\mathcal{O}^\vee)}.$$

A Harish-Chandra module X for G is called *integral special unipotent* if there exists an orbit $\mathcal{O}^\vee$ such that $\chi(\mathcal{O}^\vee)$ is integral and $\mathrm{Ann}(X) = \mathrm{J}_{\max}(\mathcal{O}^\vee)$ (with notation as in Section 2.3).

2.5. **A family of nilpotent orbits.** Suppose $\mathfrak{g} = \mathfrak{so}(2l,\mathbb{C})$. (In applications below we will take $2l = 2m + r$.) Then $\mathcal{N}$ is parametrized by partitions of $2l$ in which each even part occurs with even multiplicity. (In the case where all even parts have even multiplicity, there is an additional complication; it does not arise for us, however, and we ignore it.) Fix s even so that $2s \le 2m \le l$. We let $\mathcal{O}(s)$ denote the orbit parametrized as follows:

$$\mathcal{O}(s) = 3^s 2^{2m-2s} 1^{2l-4m+s} \qquad \text{if } (l,s) \ne (2m,0) \tag{2.3}$$

$$\mathcal{O}(s) = 2^{2m-2s-2} 1^4 \qquad \text{if } (l,s) = (2m,0); \tag{2.4}$$

Next suppose $\mathfrak{g} = \mathfrak{so}(2l+1,\mathbb{C})$. (In applications below we will take $2l+1 = 2m+r$.) Then $\mathcal{N}$ is again parametrized by partitions of $2l+1$ in which each even part occurs with even multiplicity. Fix

s odd so that $2s \leq 2m \leq l$. We let $\mathcal{O}(s)$ denote the orbit parametrized as follows:

$$\mathcal{O}(s) = 3^s 2^{2m-2s} 1^{2l-4m+s+1}. \tag{2.5}$$

2.6. **Explicit details of the representation π_s': even orthogonal groups.** Fix integers s, m, and l such that s is even, $m \leq l/2$, and $0 \leq s \leq m$. Recall the definition of π_s' given in the introduction. In the notation of Sections 2.4 and 2.5, set $\mathcal{O}^\vee(s) = d(\mathcal{O}(s))$. The main result [29, Theorem 1.1] shows that π_s' is special unipotent attached to $\mathcal{O}^\vee(s)$. More precisely,

$$\mathrm{Ann}(\pi_s') = \mathrm{J}_{\max}(\mathcal{O}^\vee(s)) \text{ and } \mathrm{AV}(\pi_s') = \overline{\mathcal{O}(s)}; \tag{2.6}$$

here we are using the notation of Section 2.3.

We recall the infinitesimal character ν_s of π_s'. In the usual coordinates we have,

$$\nu_s = \chi(\mathcal{O}^\vee(s)) = (\overbrace{0, 1, \ldots, l-m-1}^{l-m}, \overbrace{\left|\frac{s}{2} - m\right|, \left|\frac{s}{2} - m + 1\right|, \ldots, \left|\frac{s}{2} - 1\right|}^{m}). \tag{2.7}$$

(The use of ν_s to represent both an infinitesimal character and a particular representative causes no confusion in practice.) Notice ν_s is integral since s is even.

Finally we recall the lowest K-type of π_s'. Retain the notation of the introduction and, in particular, let λ_s denote the differential of the character $\det^{-l+\frac{s}{2}} \otimes \mathbf{1}$ of $L_{\mathbb{R}}$. Our hypotheses guarantee that

$$\Lambda_s = \lambda_s + 2\rho(\mathfrak{u} \cap \mathfrak{p}) = (l - 2m + \tfrac{s}{2}, \ldots, l - 2m + \tfrac{s}{2}; 0, \ldots, 0) \tag{2.8}$$

is dominant, and hence parametrizes (cf. Section 2.9) the lowest K-type in π_s. Therefore, by definition, Λ_s is the lowest K-type of π_s'

2.7. **Explicit details of the representation π_s': odd orthogonal groups.** Fix integers s, m, and l such that s is odd, $m \leq l/2$, and $0 \leq s \leq m$. In the notation of Section 2.5 and 2.4, set $\mathcal{O}^\vee(s) = d(\mathcal{O}(s))$. Again [29, Theorem 1.1] shows that π_s' is special unipotent attached to $\mathcal{O}(s)$, and the conclusions of Equation (2.6) again hold. We recall the infinitesimal character ν_s of π_s'. In the usual coordinates we have,

$$\nu_s = \chi(\mathcal{O}^\vee(s)) = (|-1 + \tfrac{s}{2}|, |-2 + \tfrac{s}{2}|, \ldots, |-m + \tfrac{s}{2}|; l - m - \tfrac{1}{2}, l - m - \tfrac{3}{2}, \ldots, \tfrac{1}{2}). \tag{2.9}$$

Notice ν_s is integral since s is odd.

Finally we recall the lowest K-type of π_s and π_s'. It is parametrized (cf. Section 2.9) by

$$\Lambda_s = \lambda_s + 2\rho(\mathfrak{u} \cap \mathfrak{p}) = (l - 2m + 1 + \frac{1}{2}(s-1), \ldots, l - 2m + 1 + \frac{1}{2}(s-1); 0, \ldots, 0). \tag{2.10}$$

2.8. **Two auxiliary representations.** The discussion in Sections 2.6 and 2.7 made use of the fact that $m \leq l/2$. We now relax that condition and assume only that $m \leq l$. The definition π_s of Equation (1.1) still makes sense. The module π_s is in the weakly fair range if and only if $s \geq m+1$. Now take $s = m$ or $m+2$ and assume m has the same parity as l. So π_m and π_{m+2} "straddle" the weakly fair range as discussed in the introduction. It is easy to check that they have the same infinitesimal character and associated variety. In fact, the methods of [29] show that π_m and π_{m+2} are both irreducible and special unipotent attached to $\mathcal{O}^\vee(m)$ where $\mathcal{O}^\vee(m) = d(\mathcal{O}(m))$ and $\mathcal{O}(m)$ is defined by the explicit partitions given in Section 2.5 (which still make sense even though m is now only assumed to be weakly less than l). Notice also that the equations defining Λ_m and Λ_{m+2} in (2.8) and (2.10) still give a dominant K-type.

2.9. **Representations of $O(p,q)$ and $SO_e(p,q)$, and K-types.** If p and q are positive integers, let $p_0 = [\frac{p}{2}]$ and $q_0 = [\frac{q}{2}]$. We list the irreducible representations of $O(p)$ by parameters $\mu = (\mu_0; \epsilon)$, where $\mu_0 = (a_1, a_2, \dots, a_{p_0})$ with the a_i non-increasing non-negative integers, and $\epsilon = \pm 1$, as described in §2.2 of [24]. The $O(n)$-type parametrized by $(\mu_0; -\epsilon)$ is obtained from that given by $(\mu_0; \epsilon)$ by tensoring with the determinant character, and the parameters $(\mu_0; \epsilon)$ and $(\mu_0; -\epsilon)$ correspond to the same representations of $O(p)$ if and only if p is even and $a_{p_0} > 0$. In this case, the restriction to $SO(p)$ is the sum of two representations with highest weights μ_0 and $(a_1, a_2, \dots, a_{p_0-1}, -a_{p_0})$, respectively. In all other cases, the restriction to $SO(p)$ is irreducible and has highest weight μ_0. A K-type for $O(p,q)$ can be specified by a parameter of the form $(a_1, a_2, \dots, a_{p_0}; \epsilon) \otimes (b_1, \dots, b_{q_0}; \eta)$ with the a_i and b_i non-decreasing non-negative integers, and $\epsilon, \eta = \pm 1$. In many cases in this paper, the signs (ϵ, η) may be chosen to be $(+1, +1)$; in that case we will often omit them and simply write the K-type by giving its highest weight $(a_1, a_2, \dots, a_{p_0}; b_1, \dots, b_{q_0})$.

The irreducible admissible representations of $O(p,q)$ may be obtained by induction from irreducible admissible representations of the identity component $SO_e(p,q)$; since $SO_e(p,q)$ has index four in $O(p,q)$, the resulting induced representation can have one, two, or four irreducible summands, resulting in four, two, or one non-equivalent irreducible admissible representations of $O(p,q)$ containing a given representation of $SO_e(p,q)$ as a summand in its restriction. (Two such representations differ by tensoring with one of the one-dimensional representations of $O(p,q)$; see §3.2 of [24] for more details.) For our representations π'_s, we are typically in the intermediate situation: there are two non-equivalent representations of $O(2m,r)$ (distinguished by signs as indicated in §5), each having two summands when restricted to the identity component, one of which is π'_s. The second summand is a representation of $SO_e(2m,r)$ whose lowest K-type is obtained from Λ_s by changing the sign of the mth entry. Only one of the two representations of $O(2m,r)$ occurs as a stable range theta lift.

3. The Correspondence and K-Types

We start by recalling the correspondence of K-types in the space of joint harmonics $\mathcal{H}$ for the dual pairs $\big(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(p,q)\big)$ [10]. We identify K-types for $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ (i. e., irreducible representations of $\widetilde{\mathrm{U}}(n)$) with their highest weights, and K-types for $O(p,q)$ as described in §2.9. Recall that each K-type μ which occurs in the Fock space $\mathcal{F}$ of the oscillator representation has associated to it a degree (the minimum degree of polynomials in the μ-isotypic subspace), and that if π and π' are representations of $\widetilde{G}$ and $\widetilde{G}'$, respectively, which correspond to each other, then each K-type for $\widetilde{G}$ which is of minimal degree in π will occur in $\mathcal{H}$ and correspond to a K-type for $\widetilde{G}'$ of minimal degree in π'. Since for a given choice of n, $\mathrm{Sp}(2n,\mathbb{R})$ is a member of many dual pairs, we refer to the degree of a K-type σ for $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ for the dual pair $\big(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(p,q)\big)$ as the (p,q)-*degree* of σ. As we will see below in Proposition 3.1, the degree of a K-type for $\mathrm{Sp}(2n,\mathbb{R})$ depends on the difference $p-q$ only, and the degree of a K-type for $\mathrm{O}(p,q)$ is independent of n. However, it depends not only on the highest weight but also on the signs. Consequently, two K-types with the same highest weight but different signs may have different degrees.

Proposition 3.1. *Let p, q, and n be non-negative integers, $p_0 = [\frac{p}{2}]$, and $q_0 = [\frac{q}{2}]$. The correspondence of K-types in the space of joint harmonics $\mathcal{H}$ for the dual pair $\big(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(p,q)\big)$ is given as follows.*

(1) *Let*

$$\mu = (a_1, a_2, \dots, a_x, 0, \dots, 0; \epsilon) \otimes (b_1, b_2, \dots, b_y, 0, \dots, 0; \eta) \tag{3.2}$$

be a K-type for $\mathrm{O}(p,q)$, with $a_x > 0$ and $b_y > 0$. Then μ occurs in $\mathcal{H}$ if and only if $n \geq x + \frac{1-\epsilon}{2}(p-2x) + \frac{1-\eta}{2}(q-2y) + y$. In that case, μ corresponds to

$$\left(\tfrac{p-q}{2}, \tfrac{p-q}{2}, \dots, \tfrac{p-q}{2}\right) + (a_1, \dots, a_x, \underbrace{1, \dots, 1}_{\frac{1-\epsilon}{2}(p-2x)}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{\frac{1-\eta}{2}(q-2y)}, -b_y, \dots, -b_1). \tag{3.3}$$

(2) *If a K-type μ for $\mathrm{O}(p,q)$ as in (3.2) occurs in the Fock space, then the degree of μ is*

$$\sum_{i=1}^{x} a_i + \sum_{i=1}^{y} b_i + \frac{1-\epsilon}{2}(p-2x) + \frac{1-\eta}{2}(q-2y). \tag{3.4}$$

For a K-type for $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ ξ which occurs in $\mathcal{F}$, write

$$\xi = \left(\tfrac{p-q}{2}, \tfrac{p-q}{2}, \dots, \tfrac{p-q}{2}\right) + (a_1, a_2, \dots, a_n). \tag{3.5}$$

Then the (p,q)-degree of ξ is $\sum_{i=1}^{n} |a_i|$.

Proof. This is well known and follows easily from the results of [16] and [10]. $\square$

In many cases it turns out that the lowest K-type of a representation which occurs in the correspondence turns out to have minimal degree in the above sense. This is true in particular for a representation of the member of the dual pair which has greater rank.

Proposition 3.6. *Let p, q, and n be non-negative integers, and suppose π and π' are genuine irreducible admissible representations of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ and $\widetilde{\mathrm{O}}(p,q)$ which correspond to each other in the correspondence for the dual pair $\big(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(p,q)\big)$.*

1. *If $p+q \le 2n+1$ then every lowest K-type of π is of minimal (p,q)-degree in π.*
2. *Let $2n+1 \le p+q$, and let Λ_0 be the highest weight of a lowest K-type of π'. Then there exists a lowest K-type Λ of π' with highest weight Λ_0 such that Λ is of minimal degree in π'.*

Proof. For the case $p+q$ even this is Corollary 37 of [24]. If $p+q = 2n+1$ this follows from Corollary 5.2 of [1]. (In fact, here all lowest K-types of π' are of minimal degree in π'.) If $p+q$ is odd and $p+q < 2n+1$ let $k = \frac{1}{2}(2n+1-p-q)$ so that $(p+k)+(q+k) = 2n+1$. We know by the persistence principle (due to Kudla; this also follows from the induction principle, Theorem 8.4 of [1]) that π occurs in the correspondence for the dual pair $\big(Sp(2n,\mathbb{R}), O(p+k,q+k)\big)$, so that the lowest K-types of π are of minimal $(p+k,q+k)$-degree. Since the (p,q)-degree and the $(p+k,q+k)$-degree of a K-type for $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ coincide, the lowest K-types of π are of minimal degree for the original dual pair. The case $p+q > 2n+1$ is analogous. $\square$

Now we compute lowest K-types for our double stable range lifts; they turn out to be unique up to signs. (The conclusion of Proposition 3.7 may be extracted from the techniques and results of [22] or [20]. We give an alternative approach.)

Proposition 3.7. *Let p, q, m, and r be non-negative integers such that $p+q \le m \le \frac{r}{2}$. Let χ be either the trivial or determinant representation of $\mathrm{O}(p,q)$. Recall from Section 2.2 the maps*

$$\theta \,:\, \mathrm{Irr}(\mathrm{O}(p,q)) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{Sp}}(2m,\mathbb{R}))$$

and

$$\theta' \,:\, \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{Sp}}(2m,\mathbb{R})) \longrightarrow \mathrm{Irr}(\mathrm{O}(2m,r)),$$

and set $\pi = \theta(\chi)$ and $\pi' = \theta'(\pi)$.

(1) *If $\chi = \mathbb{1}$ then the K-types of π are precisely those of the form*

$$\left(\frac{p-q}{2}, \ldots, \frac{p-q}{2}\right) + (a_1, a_2, \ldots, a_p, 0, \ldots, 0, -b_q, \ldots, -b_1), \tag{3.8}$$

where a_i and b_i are non-negative even integers for all i, and each K-type occurs with multiplicity one.

(2) *If $\chi = \mathbb{1}$, let $a = \frac{p-q+r}{2} - m$. Then π' has a lowest K-type Λ, unique up to signs, given by*

$$\Lambda = \begin{cases} (a, a, \ldots, a, \underbrace{0, \ldots, 0}_{q}; 0, \ldots, 0) & \text{if } a \geq 0 \text{ and even;} \\ (a, a, \ldots, a, \underbrace{1, \ldots, 1}_{q}; 0, \ldots, 0) & \text{if } a > 0 \text{ and odd;} \\ (0, \ldots, 0; \underbrace{-a, \ldots, -a}_{m-p}, 0, \ldots, 0) & \text{if } a \leq 0 \text{ and even;} \\ (0, \ldots, 0; \underbrace{-a, \ldots, -a}_{m-p}, \underbrace{1, \ldots, 1}_{p}, 0, \ldots, 0) & \text{if } a < 0 \text{ and odd.} \end{cases} \tag{3.9}$$

(3) *If $\chi = det$ then the K-types of π are precisely those of the form (3.8) with a_i and b_i positive odd integers for all i, and each K-type occurs with multiplicity one.*

(4) *If $\chi = det$, let $a = \frac{p-q+r}{2} - m$. Then π' has a lowest K-type Λ, unique up to signs, given by*

$$\Lambda = \begin{cases} (\underbrace{1, \ldots, 1}_{p}, 0, \ldots, 0; \underbrace{1, \ldots, 1}_{q}, 0, \ldots, 0) & \text{if } a = 0\ ; \\ (\underbrace{a+1, \ldots, a+1}_{p}, a, \ldots, a, \underbrace{0, \ldots, 0}_{q}; 0, \ldots, 0) & \text{if } a > 0 \text{ and odd;} \\ (\underbrace{a+1, \ldots, a+1}_{p}, a, \ldots, a, \underbrace{1, \ldots, 1}_{q}; 0, \ldots, 0) & \text{if } a > 0 \text{ and even;} \\ (0, \ldots, 0; \underbrace{-a+1, \ldots, -a+1}_{q}, \underbrace{-a, \ldots, -a}_{m-p-q}, 0, \ldots, 0) & \text{if } a < 0 \text{ and odd;} \\ (0, \ldots, 0; \underbrace{-a+1, \ldots, -a+1}_{q}, \underbrace{-a, \ldots, -a}_{m-p-q}, \underbrace{1, \ldots, 1}_{p}, 0, \ldots, 0) & \text{if } a < 0 \text{ and even.} \end{cases} \tag{3.10}$$

Proof. The first part of the proposition is Proposition 2.1, together with Corollary 2.7(c), of [17]. For (2), let

$$\sigma = \left(\tfrac{p-q}{2}, \ldots, \tfrac{p-q}{2}\right) + (a_1, a_2, \ldots, a_p, 0, \ldots, 0, -b_q, \ldots, -b_1), \tag{3.11}$$

be a K-type of π. Then

$$\sigma = \left(\tfrac{2m-r}{2}, \ldots, \tfrac{2m-r}{2}\right) + (a + a_1, \ldots, a + a_p, a, \ldots, a, a - b_q, \ldots, a - b_1), \tag{3.12}$$

and the $(2m, r)$-degree of σ is

$$d = \sum_{i=1}^{p} |a + a_i| + \sum_{i=1}^{q} |a - b_q| + (m - p - q)|a|. \tag{3.13}$$

Suppose a is even. Then the degree of σ is minimized if $a_i = 0$ and $b_i = a$ for all i if $a \geq 0$, and by choosing $a_i = -a$ and $b_i = 0$ for all i if a is negative. So π has a unique K-type σ_0 of minimal

$(2m,r)$-degree. The K-type Λ corresponding to σ_0 in the space of joint harmonics for the dual pair $\big(Sp(2m,\mathbb{R}),O(2m,r)\big)$ (see Proposition 3.1) must then be a lowest K-type of π', unique up to signs, by Proposition 3.6.

If a is odd then π has more than one K-type of minimal $(2m,r)$-degree: for $a<0$ choose $b_i=0$ for all i, and $a_i=-a+1$ or $-a-1$ in such a way that the resulting weight is dominant. The corresponding K-types for $O(2m,r)$ are then those of the form

$$\Lambda_{k,t}=\big(\underbrace{1,\dots,1}_{k},0,\dots,0;\underbrace{-a,\dots,-a}_{m-p},\underbrace{1,\dots,1}_{t},0,\dots,0\big) \tag{3.14}$$

with $k+t=p$. By Proposition 3.6, the lowest K-types of π' are precisely those $\Lambda_{k,t}$ for which the Vogan-norm ([33] Definition 5.4.18) of $\Lambda_{k,t}$,

$$||\Lambda_{k,t}||=<\Lambda_{k,t}+2\rho_c,\Lambda_{k,t}+2\rho_c> \tag{3.15}$$

is minimal. Since the quantity $<\Lambda_{k,t},\Lambda_{k,t}>$ only depends on $k+t=p$, we can minimize the quantity $n_{k,t}=\frac{1}{2}\big(||\Lambda_{k,t}||-<2\rho_c,2\rho_c>-<\Lambda_{k,t},\Lambda_{k,t}>\big)$ instead. If $n=[\frac{r}{2}]$ then $2\rho_c=(2m-2,2m-4,\dots,2,0;r-2,r-4,\dots,r-2n)$. So

$$n_{k,t}=\sum_{i=1}^{k}(2m-2i)+\sum_{i=1}^{m-p}(-a)(r-2i)+\sum_{i=1}^{t}(r-2m+2p-2i). \tag{3.16}$$

The second sum is independent of k and t, and it is clear that $n_{k,t}$ will be minimized by $k=p$, $t=0$ if $2m\le r-2m+2p$, and by $k=0$, $t=p$ if $2m\ge r-2m+2p$. It remains to show that we must have $4m>r+2p$. We know that $a=\frac{p-q+r}{2}-m<0$, so $p-q+r<2m$, Moreover, we have assumed that $p+q\le m$, so we get $2p+r<3m\le 4m$, and we are done with this case. The case $a>0$ is much easier and left to the reader.

Part (3) is Proposition 2.4 of [18], together with Proposition 2.1 of [11], and part (4) can be easily obtained using (3) and an argument similar to the one used for the case $\chi=\mathbb{1}$. We omit the details. $\square$

We conclude by introducing some additional notation. Fix integers $s\le m\le r/2$ so that the parities of s and r match. Recall the double lift θ^2 of Section 2.2 and assume that the choices made defining θ^2 match those in Proposition 3.7. Consider the set of highest weights of lowest K-types of the restriction of $\theta^2(\mathbf{1}_s)$ to $\mathrm{SO}_e(2m,r)$. In terms of the parametrization of Section 2.9, this set contains a unique highest weight (possibly occurring with multiplicity greater than one) whose coordinates are all nonnegative. Call the corresponding lowest K-type "positive". Then set

$$\Lambda_s=\text{the } K\text{-type that occurs as a "positive" lowest } K\text{-type in the restriction of } \theta^2(\mathbf{1}_s) \text{ to } \mathrm{SO}_e(2m,r) \tag{3.17}$$

and

$$\Lambda'_s=\text{the } K\text{-type that occurs as a "positive" lowest } K\text{-type in the restriction of } \theta^2(\det_s) \text{ to } \mathrm{SO}_e(2m,r). \tag{3.18}$$

Explicit formulas for them are given in Proposition 3.7. Notice that Λ_s has already been defined in Sections 2.6 and 2.7. Using Proposition 3.7, it is easy to check that the two definitions coincide.

4. Characterizing certain unipotent representations by their lowest K types: proof of Theorem 1.2 and Proposition 1.3

As alluded to in the introduction, the following is the key uniqueness result we need. It allows us to identify our special unipotent representations of interest by simply computing their associated varieties, infinitesimal characters, and lowest K-types.

Proposition 4.1. *Recall the notation of Sections 2.6 and 2.7, as well as that of Equations (3.17) and (3.18). There is a unique special unipotent representation of* $\mathrm{SO}_e(2m,r)$ *attached to* $\mathcal{O}^\vee(s)$ *with lowest* K*-type* Λ_s*. Similarly if* $s \neq 0$ *there is a unique special unipotent representation attached to* $\mathcal{O}^\vee(s)$ *with lowest* K*-type* Λ'_s*.*

Remark 4.2. The proof is rather technical, and we defer it to Section 5. Recall (from the discussion in Section 2.4 for instance) that a representation is a special unipotent representation if it has the right infinitesimal character and size. So the proposition roughly says that the infinitesimal character, size, and lowest K-types characterize certain representations uniquely. For many of the representations appearing in the proposition, the infinitesimal character and lowest K-types are enough to guarantee unicity; but there are some for which this is not enough and the further consideration of size must also be invoked.

As we shall see, the proof also extends to another case which will be important for us. Retain the relaxed setting of Section 2.8. Then π_m is the unique special unipotent representation attached to $\mathcal{O}^\vee(m)$ with lowest K-type Λ_m. Likewise π_{m+2} is the unique special unipotent representation attached to $\mathcal{O}^\vee(m)$ with lowest K type Λ_{m+2}. □

The next proposition says that the iterated lifts $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ from $\mathrm{O}(s,0)$ to $\mathrm{Sp}(2m,\mathbb{R})$ to $\mathrm{O}(2m,r)$ contain the unique representations of Proposition 4.1.

Proposition 4.3. *Fix integers* $s \leq m \leq r/2$ *so that the parity of* s *coincides with that of* r*. The double lift* $\theta^2(\mathbf{1}_s)$ *of the trivial representation lifted from* $\mathrm{O}(s)$ *to* $\mathrm{Sp}(2m,\mathbb{R})$ *and then to* $\mathrm{O}(2m,r)$ *restricted to* $\mathrm{SO}_e(2m,r)$ *contains the unique special unipotent representation of* $\mathrm{SO}_e(2m,r)$ *attached to* $\mathcal{O}^\vee(s)$ *with lowest* K*-type* Λ_s*. Similarly if* $s \neq 0$ *the restriction of the double lift* $\theta^2(\det_s)$ *of the determinant representation contains the unique special unipotent representation of* $\mathrm{SO}_e(2m,r)$ *attached to* $\mathcal{O}^\vee(s)$ *with lowest* K*-type* Λ'_s*.*

Proof. That Λ_s and Λ'_s are the indicated lowest K-types follows from the definitions of Equations (3.17) and (3.18). So all that remains to show is that the two double lifts are special unipotent attached to $\mathcal{O}^\vee(s)$. Using the correspondence of infinitesimal characters ([27]), it is easy to check that the two double lifts have the required infinitesimal character ν_s given in Sections 2.6 and 2.7. Hence it remains to verify only that the dense orbit (say $\mathcal{O}_s$) in the associated variety of the annihilator of the double lifts is indeed $\mathcal{O}(s)$. Since there are no representations with infinitesimal character ν_s with smaller associated variety (as follows from the general theory of special unipotent representations), it enough to show that that $\mathcal{O}_s \subset \mathcal{O}(s)$. Now the paper [26] gives an explicit upper bound on $\mathcal{O}_s$ in terms of certain moment map images. That upper bound can be computed explicitly and shown to coincide with $\mathcal{O}(s)$. (Computations of this sort are explained very carefully in [31].) The proof is complete. □

The main results of the introduction are now simple corollaries.

Proof of Theorem 1.2. As explained in Sections 2.6 and 2.7, [29] shows that π'_s is special unipotent attached to $\mathcal{O}(s)$ with lowest K-type Λ_s. So the theorem follows from Propositions 4.1 and 4.3.

Proof of Proposition 1.3. In the setting of Proposition 1.3, [29] shows that π_m and π_{m+2} are special unipotent representations attached to $\mathcal{O}^\vee(m)$ with respective lowest K-types Λ_m and Λ'_m. So the current proposition follows from Propositions 4.1 and 4.3. □

We now explain how to extend Proposition 1.3 to a more general setting. The idea, as mentioned in the introduction, is to induce $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ to special unipotent representations of a larger group G' that we can quickly recognize in terms of cohomological induction. The subtlety is to arrange the induction so that we indeed obtain (nonzero) special unipotent representations of G'. Here are the details.

Again fix integers $s \leq m \leq r/2$ so that the parity of s matches that of r. Consider the restrictions to $\mathrm{SO}_e(2m, r)$, say $\theta_e^2(\mathbf{1}_s)$ and $\theta_e^2(\det_s)$, of the double lifts $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ which contain the respective lowest K-types Λ_s and Λ_s'. Let $G' = \mathrm{SO}_e(4m - 2s, r)$ and maintain the notation of Section 2.1 with the addition of primes as appropriate. Let $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ be a τ'-stable parabolic of $\mathfrak{g}'$ so that $\mathfrak{l}' \cap \bar{\mathfrak{l}}'$ corresponds to

$$L' = \mathrm{U}(m - s, 0) \times \mathrm{SO}_e(2m, r).$$

Suppose that $\mathfrak{u}'$ corresponds to a choice of positive roots so that the roots corresponding to the $\mathrm{U}(m - s, 0)$ factor appear before those in the SO factor. Consider the $(\mathfrak{l}', L' \cap K')$ module

$$\det^k \otimes \theta_e^2(\mathbf{1}_s),$$

where

$$k = -l + [s/2],$$

and $[s/2]$ is the greatest integer less than $s/2$. One may verify that $\det^k \otimes \theta_e^2(\mathbf{1}_s)$ is in the weakly fair range for $\mathfrak{q}$ ([15, Definition 0.35]). Let S be the middle degree $\dim(\mathfrak{u}' \cap \mathfrak{k}')$, and finally consider the derived functor module

$$\Gamma_{\mathbf{1}} = \mathcal{R}_{\mathfrak{q}'}^S(\det^k \otimes \theta_e^2(\mathbf{1}_s)). \tag{4.4}$$

Again we follow the normalization of [15] (and the notation of [33]) so that there is a ρ-shift in the infinitesimal character of Γ_1. Using the explicit formula for the lowest K-type Λ_s of $\theta_e^2(\mathbf{1}_s)$ given above, it is simple to check that the highest weight of the lowest K-type of $\det^k \otimes \theta_e^2(\mathbf{1}_s)$ shifted by $2\rho(\mathfrak{u}' \cap \mathfrak{p}')$ is still dominant and hence parametrizes the lowest K-type of Γ_1 (which, in particular, is thus nonzero). It seems likely that the methods of [35] could be applied to show that Γ_1 is irreducible. To be on the safe side, let Γ_1' denote the lowest K-type constituent of Γ_1. Similarly put

$$\Gamma_{\det} = \mathcal{R}_{\mathfrak{q}'}^S(\det^k \otimes \theta_e^2(\det_s)), \tag{4.5}$$

and let $\Gamma_{\det}'$ denote its lowest K-type constituent. (Again it is likely that $\Gamma_{\det}' = \Gamma_{\det}$.)

The following generalization of Proposition 1.3 states that the modules $\Gamma_{\mathbf{1}}'$ and $\Gamma_{\det}'$ induced from the double lifts of the trivial and determinant representation are special unipotent $A_{\mathfrak{q}}(\lambda)$ modules that straddle the weakly fair range.

Proposition 4.6. *Fix integers $s \leq m \leq r/2$ so that the parity of r matches that of s. Set $G' = \mathrm{SO}_e(4m - 2s, r)$. Consider the representations $\Gamma_{\mathbf{1}}'$ and $\Gamma_{\det}'$ of G' defined around Equations (4.4) and (4.5) above as cohomologically induced from the iterated lifts of the trivial and determinant representation of $\mathrm{O}(s, 0)$ to $\mathrm{Sp}(2m, \mathbb{R})$ to $\mathrm{O}(2m, r)$. Next recall the special unipotent representations π_{4m-2s} and $\pi_{4m-2s+2}$ of G' attached to $\mathcal{O}^\vee(4m - 2s)$ discussed in Section 2.8. (These are cohomologically induced modules that straddle the edge of weakly fair range.) Then*

$$\Gamma_{\mathbf{1}}' = \pi_{4m-2s},$$

and

$$\Gamma_{\det}' = \pi_{4m-2s+2}.$$

In particular, when $s = m$ we recover Proposition 1.3.

Proof. We first show $\Gamma_{\mathbf{1}}' = \pi_{4m-2s}$. In the discussion around the definition of $\Gamma_{\mathbf{1}}'$, we mentioned that its lowest K-type is the lowest K-type of $\det^k \otimes \theta_e^2(\mathbf{1}_s)$ shifted by $2\rho(\mathfrak{u}' \cap \mathfrak{p}')$. It is easy to check that it matches the lowest K-type Λ_{4m-2s} of π_{4m-2s} described explicitly in Equation (2.8). As mentioned in Section 2.8, π_{4m-2s} is special unipotent attached to $\mathcal{O}^\vee(4m - 2s)$. By the unicity discussed in Remark 4.2, it remains only to show that $\Gamma_{\mathbf{1}}'$ is special unipotent attached to $\mathcal{O}^\vee(4m-2s)$. It's easy to check that the infinitesimal character of $\Gamma_{\mathbf{1}}'$ matches that attached to $\mathcal{O}^\vee(4m - 2s)$. So, just as in the proof of Proposition 1.3, it suffices to check that the dense orbit in the associated variety of the annihilator of $\Gamma_{\mathbf{1}}'$, say $\mathcal{O}$, matches $\mathcal{O}(4m-2s)$. Using the main results of [30], it is not difficult in fact to compute the associated variety of $\Gamma_{\mathbf{1}}'$ given the computation of the associated variety of $\theta_e^2(\mathbf{1}_s)$, which is known by combining Theorem 1.2 (identifying $\theta_e^2(\mathbf{1}_s)$ as a Knapp representation) and [29] (computing the associated variety of Knapp representations). One finds that indeed the associated variety of $\Gamma'_{\mathbf{1}}$ is a component of $\mathcal{O} \cap \mathfrak{p}'$. So indeed $\mathcal{O} = \mathcal{O}(4m-2s)$. Thus $\Gamma'_{\mathbf{1}} = \pi_{4m-2s}$.

The proof that $\Gamma'_{\det} = \pi_{4m-2s+2}$ follows in nearly the identical way. (A mild complication is that we have yet to compute the associated variety of $\theta_e^2(\det_s)$. It coincides with that of $\theta_e^2(\mathbf{1}_s)$, but we omit the details of that calculation here.) $\square$

5. Langlands parameters and the proof of Proposition 4.1

In this section we prove Proposition 4.1. The Langlands parameters of the double lifts $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ are given in Theorem 5.33.

We start by defining representations $\pi_{\mathbb{1}}$ and $\pi_{\det}$ of $\mathrm{O}(2m, r)$ which will turn out to be the double theta lifts of the trivial and determinant representations of $\mathrm{O}(s, 0)$, respectively, by giving their Langlands parameters.

We use the notation established in [24]; following Vogan's version of the Langlands Classification [34], a set of Langlands parameters of an irreducible admissible representation π of $\mathrm{O}(p, q)$ consists of a Levi subgroup $MA \cong \mathrm{O}(p-2t-k, q-2t-k) \times \mathrm{GL}(2, \mathbb{R})^t \times \mathrm{GL}(1, \mathbb{R})^k$ of $\mathrm{O}(p, q)$ and data $(\lambda, \Psi, \mu, \nu, \epsilon, \kappa)$ with (λ, Ψ) the Harish-Chandra parameter and system of positive roots determining a limit of discrete series ρ of $\mathrm{O}(p-2t-k, q-2t-k)$, $\mu \in \mathbb{Z}^t$ and $\nu \in \mathbb{C}^t$ determining a relative limit of discrete series τ of $\mathrm{GL}(2, \mathbb{R})^t$, and the pair (ϵ, κ) with $\epsilon \in \{\pm 1\}^k$ and $\kappa \in \mathbb{C}^k$ determining a character χ of $\mathrm{GL}(1, \mathbb{R})^k$. (The group MA is of course implied by the other data.) The representation $\pi = \pi(\lambda, \Psi, \mu, \nu, \epsilon, \kappa)$ is then an irreducible quotient of an induced representation $\mathrm{Ind}_{MAN}^{\mathrm{O}(p,q)}(\rho \otimes \tau \otimes \chi \otimes \mathbb{1})$, with $P = MAN$ chosen such that certain positivity conditions are satisfied. Since $\mathrm{O}(p, q)$ is disconnected, there may be more than one such irreducible quotient which can be distinguished by signs (see §3.2 of [24]). Here, we will always mean the representation with all signs positive, so we omit them.

Let m, r, and s be integers as before, i. e., $m \geq 1$, $r \geq 2m$, $0 \leq s \leq m$, and $s \equiv r (mod\, 2)$. We define $\pi_{\mathbb{1}}$ to be the representation of $\mathrm{O}(2m, r)$ with the following Langlands parameters:

(1) If $r = 2m$ and $s = 0$ then $\pi_{\mathbb{1}}$ is the spherical representation with $MA \cong \mathrm{GL}(1, \mathbb{R})^{2m}$, given by $\pi_{\mathbb{1}} = \pi(0, \emptyset, 0, 0, \epsilon, \kappa)$, where $\epsilon = (1, \ldots, 1)$ and $\kappa = \nu_0$. (Here ν_0 is the infinitesimal character of π'_0 as in (2.7).)

(2) If $r \geq 2m+s$ let $\pi_{\mathbb{1}} = \pi(\lambda_d, \Psi, \mu, \nu, 0, 0)$ with $MA \cong \mathrm{O}(0, r-2m) \times \mathrm{GL}(2, \mathbb{R})^m$,

$$\lambda_d = \left(\frac{r}{2} - m - 1, \frac{r}{2} - m - 2, \ldots, 1, 0\right) \quad \text{or} \quad \lambda_d = \left(\frac{r}{2} - m - 1, \frac{r}{2} - m - 2, \ldots, \frac{3}{2}, \frac{1}{2}\right) \tag{5.1}$$

depending on whether r is even or odd,

$$\begin{aligned} \mu &= \left(\frac{r+s}{2} - m - 1, \frac{r+s}{2} - m - 1, \ldots, \frac{r+s}{2} - m - 1\right), \\ \nu &= \left(\frac{r-s}{2} + m - 1, \frac{r-s}{2} + m - 3, \ldots, \frac{r-s}{2} - m + 3, \frac{r-s}{2} - m + 1\right), \end{aligned} \tag{5.2}$$

and Ψ is the positive root system (uniquely) determined by λ_d.

(3) If $r \leq 2m+s$, and $r > 2m$ or $s > 0$ (so that we are not in the first case) we distinguish two cases, depending on the parity of $\frac{r-s}{2} + m$.

If $\frac{r-s}{2} + m$ is even then $\pi_{\mathbb{1}} = \pi(\lambda_d, \Psi, \mu, \nu, 0, 0)$ with $MA \cong \mathrm{O}(m - \frac{r-s}{2}, \frac{r+s}{2} - m) \times \mathrm{GL}(2, \mathbb{R})^{\frac{r-s}{4} + \frac{m}{2}}$,

$$\begin{aligned} \lambda_d &= \left(\frac{s}{2} - 1, \frac{s}{2} - 2, \ldots, \frac{r+s}{4} - \frac{m}{2}; \frac{r+s}{4} - \frac{m}{2} - 1, \frac{r+s}{4} - \frac{m}{2} - 2, \ldots, 1, 0\right) \quad \text{or} \\ \lambda_d &= \left(\frac{s}{2} - 1, \frac{s}{2} - 2, \ldots, \frac{r+s}{4} - \frac{m}{2}; \frac{r+s}{4} - \frac{m}{2} - 1, \frac{r+s}{4} - \frac{m}{2} - 2, \ldots, \frac{3}{2}, \frac{1}{2}\right) \end{aligned} \tag{5.3}$$

depending on whether r is even or odd,

$$\begin{aligned} \mu &= \left(\frac{r+s}{2} - m - 1, \frac{r+s}{2} - m - 1, \dots, \frac{r+s}{2} - m - 1\right), \\ \nu &= \left(\frac{r-s}{2} + m - 1, \frac{r-s}{2} + m - 3, \dots, 3, 1\right), \end{aligned} \tag{5.4}$$

and Ψ is the positive root system (uniquely) determined by λ_d.

If $\frac{r-s}{2} + m$ is odd then $\pi_{\mathbb{1}} = \pi(\lambda_d, \Psi, \mu, \nu, 0, 0)$ with $MA \cong \mathrm{O}(m - \frac{r-s}{2} + 1, \frac{r+s}{2} - m + 1) \times \mathrm{GL}(2, \mathbb{R})^{\frac{r-s}{4} + \frac{m}{2} - \frac{1}{2}}$,

$$\begin{aligned} \lambda_d &= \left(\frac{s}{2} - 1, \frac{s}{2} - 2, \dots, \frac{r+s}{4} - \frac{m}{2} - \frac{1}{2}; \frac{r+s}{4} - \frac{m}{2} - \frac{1}{2}, \frac{r+s}{4} - \frac{m}{2} - \frac{3}{2}, \dots, 1, 0\right) \quad \text{or} \\ \lambda_d &= \left(\frac{s}{2} - 1, \frac{s}{2} - 2, \dots, \frac{r+s}{4} - \frac{m}{2} - \frac{1}{2}; \frac{r+s}{4} - \frac{m}{2} - \frac{1}{2}, \frac{r+s}{4} - \frac{m}{2} - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}\right) \end{aligned} \tag{5.5}$$

depending on whether r is even or odd,

$$\begin{aligned} \mu &= \left(\frac{r+s}{2} - m - 1, \frac{r+s}{2} - m - 1, \dots, \frac{r+s}{2} - m - 1\right), \\ \nu &= \left(\frac{r-s}{2} + m - 1, \frac{r-s}{2} + m - 3, \dots, 4, 2\right), \end{aligned} \tag{5.6}$$

and Ψ is chosen so that the corresponding limit of discrete series of $\mathrm{O}(m - \frac{r-s}{2} + 1, \frac{r+s}{2} - m + 1)$ is holomorphic.

Now assume $s \geq 1$. We define $\pi_{\det}$ to be the representation of $\mathrm{O}(2m, r)$ with the following Langlands parameters:

(1) If $r \geq 2m + s + 2$ let $\pi_{\det} = \pi(\lambda_d, \Psi, \mu, \nu, 0, 0)$ with MA, λ_d, and Ψ as in the corresponding case for $\pi_{\mathbb{1}}$,

$$\begin{aligned} &\mu = \Big(\underbrace{\frac{r+s}{2} - m, \dots, \frac{r+s}{2} - m}_{s \text{ entries}}, \underbrace{\frac{r+s}{2} - m - 1, \dots, \frac{r+s}{2} - m - 1}_{m-s \text{ entries}}\Big), \\ &\text{and} \\ &\nu = \Big(\frac{r+3s}{2} - m - 2, \frac{r+3s}{2} - m - 4, \dots, \frac{r-s}{2} - m, \\ &\qquad \frac{r-s}{2} + m - 1, \frac{r-s}{2} + m - 3, \dots, \frac{r-s}{2} - m + 2s + 1\Big). \end{aligned} \tag{5.7}$$

(2) If $r \leq 2m + s + 2$, we once again distinguish two cases, depending on the parity of $\frac{r-s}{2} + m$.

If $\frac{r-s}{2} + m$ is odd then $\pi_{\det} = \pi(\lambda_d, \Psi, \mu, \nu, 0, 0)$ with $MA \cong \mathrm{O}(m - \frac{r-s}{2} + 1, \frac{r+s}{2} - m + 1) \times \mathrm{GL}(2, \mathbb{R})^{\frac{r-s}{4} + \frac{m}{2} - \frac{1}{2}}$,

$$\begin{aligned} \lambda_d &= \left(\frac{s}{2}, \frac{s}{2} - 1, \dots, \frac{r+s}{4} - \frac{m}{2} + \frac{1}{2}; \frac{r+s}{4} - \frac{m}{2} - \frac{1}{2}, \frac{r+s}{4} - \frac{m}{2} - \frac{3}{2}, \dots, 1, 0\right) \quad \text{or} \\ \lambda_d &= \left(\frac{s}{2}, \frac{s}{2} - 1, \dots, \frac{r+s}{4} - \frac{m}{2} + \frac{1}{2}; \frac{r+s}{4} - \frac{m}{2} - \frac{1}{2}, \frac{r+s}{4} - \frac{m}{2} - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}\right) \end{aligned} \tag{5.8}$$

depending on whether r is even or odd,

$$\begin{aligned}
\mu &= \Big(\underbrace{\frac{r+s}{2}-m,\dots,\frac{r+s}{2}-m}_{\frac{r+3s}{4}-\frac{m}{2}-\frac{1}{2}\text{ entries}},\underbrace{\frac{r+s}{2}-m-1,\dots,\frac{r+s}{2}-m-1}_{m-s\text{ entries}}\Big),\\
\nu &= \Big(\frac{r+3s}{2}-m-2,\frac{r+3s}{2}-m-4,\dots,3,1,\\
&\qquad \frac{r-s}{2}+m-1,\frac{r-s}{2}+m-3,\dots,\frac{r-s}{2}-m+2s+1\Big),
\end{aligned} \tag{5.9}$$

and Ψ is the positive root system (uniquely) determined by λ_d.

If $\frac{r-s}{2}+m$ is even then $\pi_{\det}=\pi(\lambda_d,\Psi,\mu,\nu,0,0)$ with $MA\cong \mathrm{O}(m-\frac{r-s}{2}+2,\frac{r+s}{2}-m+2)\times \mathrm{GL}(2,\mathbb{R})^{\frac{r-s}{4}+\frac{m}{2}-1}$,

$$\begin{aligned}
\lambda_d &= \Big(\frac{s}{2},\frac{s}{2}-1,\dots,\frac{r+s}{4}-\frac{m}{2};\frac{r+s}{4}-\frac{m}{2},\frac{r+s}{4}-\frac{m}{2}-1,\dots,1,0\Big) \quad \text{or}\\
\lambda_d &= \Big(\frac{s}{2},\frac{s}{2}-1,\dots,\frac{r+s}{4}-\frac{m}{2};\frac{r+s}{4}-\frac{m}{2},\frac{r+s}{4}-\frac{m}{2}-1,\dots,\frac{3}{2},\frac{1}{2}\Big)
\end{aligned} \tag{5.10}$$

depending on whether r is even or odd,

$$\begin{aligned}
\mu &= \Big(\underbrace{\frac{r+s}{2}-m,\dots,\frac{r+s}{2}-m}_{\frac{r+3s}{4}-\frac{m}{2}-1\text{ entries}},\underbrace{\frac{r+s}{2}-m-1,\dots,\frac{r+s}{2}-m-1}_{m-s\text{ entries}}\Big),\\
\nu &= \Big(\frac{r+3s}{2}-m-2,\frac{r+3s}{2}-m-4,\dots,4,2,\\
&\qquad \frac{r-s}{2}+m-1,\frac{r-s}{2}+m-3,\dots,\frac{r-s}{2}-m+2s+1\Big),
\end{aligned} \tag{5.11}$$

and Ψ is chosen so that the corresponding limit of discrete series of $\mathrm{O}(m-\frac{r-s}{2}+1,\frac{r+s}{2}-m+1)$ is holomorphic.

Recall the $\mathrm{SO}(2m)\times\mathrm{SO}(r)$-types Λ_s and Λ_s' defined in (3.17) and (3.18). We use the same notation (Λ_s and Λ_s') for the unique $\mathrm{O}(2m)\times\mathrm{O}(r)$-types with positive signs containing Λ_s and Λ_s' respectively in their restrictions to $\mathrm{SO}(2m)\times\mathrm{SO}(r)$.

Proposition 5.12. *Let m, r, s, $\pi_{\mathbb{1}}$ and $\pi_{\det}$ be as above. Then $\pi_{\mathbb{1}}$ and $\pi_{\det}$ both have infinitesimal character ν_s (see (2.7)), $\pi_{\mathbb{1}}$ has lowest K-type Λ_s, and $\pi_{\det}$ has lowest K-type Λ_s'.*

Proof. This amounts to a case-by-case calculation using the theory of [33] and [15] as described in detail and explicitly in §3.2 of [24] for r even (the odd case is very similar). If $\pi=\pi(\lambda_d,\Psi,\mu,\nu,\epsilon,\kappa)$ with $MA\cong \mathrm{O}(2m-2t-k,r-2t-k)\times\mathrm{GL}(2,\mathbb{R})^t\times\mathrm{GL}(1,\mathbb{R})^k$, write $p=[m-t-\frac{k}{2}]$, $q=[\frac{r-2t-k}{2}]$, $\lambda_d=(a_1,\dots,a_p;b_1,\dots,b_q)$, $\mu=(\mu_1,\dots,\mu_t)$, $\nu=(n_1,\dots,n_t)$, and $\kappa=(\kappa_1,\dots,\kappa_k)$. Then the infinitesimal character of π is given by

$$\gamma=\Big(a_1,\dots,a_p,b_1,\dots,b_q,\frac{\mu_1+n_1}{2},\dots,\frac{\mu_t+n_t}{2},\frac{\mu_1-n_1}{2},\dots,\frac{\mu_t-n_t}{2},\kappa_1,\dots,\kappa_k\Big). \tag{5.13}$$

To compute the lowest K-types of π, one assigns to π the Vogan parameter λ_a, an element of $\mathfrak{t}^*$ which is essentially the discrete part of the infinitesimal character, as follows: the parameter λ_a is the element of $\mathfrak{t}^*$ which is dominant with respect to a fixed positive system of compact roots and conjugate by the compact Weyl group to

$$\left(a_1,\ldots,a_p,\frac{\mu_1}{2},\ldots,\frac{\mu_t}{2},\underbrace{0,\ldots,0}_{m-t-p};b_1,\ldots,b_q,\frac{\mu_1}{2},\ldots,\frac{\mu_t}{2},\underbrace{0,\ldots,0}_{[\frac{r}{2}]-t-q}\right);\tag{5.14}$$

the lowest K-types are then of the form

$$\lambda_a+\rho(\mathfrak{u}\cap\mathfrak{p})-\rho(\mathfrak{u}\cap\mathfrak{k})+\delta_L,\tag{5.15}$$

where $\mathfrak{q}=\mathfrak{l}\oplus\mathfrak{u}$ is the theta stable parabolic subalgebra of $\mathfrak{g}$ determined by λ_a, $\rho(\mathfrak{u}\cap\mathfrak{p})$ and $\rho(\mathfrak{u}\cap\mathfrak{k})$ are one half the sum of the positive noncompact and compact roots in $\mathfrak{u}$, respectively, and δ_L is the highest weight of a fine K-type for L, a subgroup of $\mathrm{O}(2m,r)$ corresponding to $\mathfrak{l}$. The allowed choices for δ_L are determined by Ψ and ϵ (see Proposition 10 of [24]).

We omit the detailed calculation, but illustrate using an example below.

□

Example 5.16. Let $m=s=2$ and $r=6$. Then $\pi_{\mathbb{1}}$ has Langlands parameters $MA\cong\mathrm{O}(0,2)\times\mathrm{GL}(2,\mathbb{R})^2$, $\lambda_d=(0)$, $\mu=(1,1)$, and $\nu=(3,1)$ (using either case (2) or (3)). So $\lambda_a=(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2},0)$. We have the infinitesimal character of $\pi_{\mathbb{1}}$ given by

$$\left(\frac{1}{2}+\frac{3}{2},\frac{1}{2}+\frac{1}{2},\frac{1}{2}-\frac{3}{2},\frac{1}{2}-\frac{1}{2},0\right)=(2,1,-1,0,0),\tag{5.17}$$

which is Weyl group conjugate to ν_2. The lowest K-type is of the form

$$\begin{aligned}&\lambda_a+\rho(\mathfrak{u}\cap\mathfrak{p})-\rho(\mathfrak{u}\cap\mathfrak{k})+\delta_L\\ =&\left(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2},0\right)+(2,2;1,1,0)-\left(\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2},0\right)+(0,0;0,0,0)\\ =&(2,2;0,0,0)=\Lambda_2.\end{aligned}\tag{5.18}$$

For $\pi_{\det}$ we have $MA\cong\mathrm{O}(2,4)\times\mathrm{GL}(2,\mathbb{R})$, $\lambda_d=(1;1,0)$, $\mu=(2)$ and $\nu=(2)$ (using case (2)). So $\lambda_a=(1,1;1,1,0)$, and the infinitesimal character is

$$(1+1,1,1-1,1,0)=(2,1,0,1,0)\underset{W}{\sim}\nu_2.\tag{5.19}$$

The lowest K-type is

$$\begin{aligned}&\lambda_a+\rho(\mathfrak{u}\cap\mathfrak{p})-\rho(\mathfrak{u}\cap\mathfrak{k})+\delta_L\\ =&\,(1,1;1,1,0)+(2,2;1,1,0)-\left(\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2},0\right)+\left(\frac{1}{2},\frac{1}{2};-\frac{1}{2},-\frac{1}{2},0\right)\\ =&(3,3;0,0,0)=\Lambda_2'.\end{aligned}\tag{5.20}$$

Here the fine K-type δ_L is uniquely determined by Ψ which is such that the resulting limit of discrete series on $\mathrm{O}(2,4)$ is holomorphic.

We show below that in many cases, including most cases for $\pi_{\mathbb{1}}$ and the case $s=m$ for $\pi_{\det}$, this representation is uniquely determined by its lowest K-type and infinitesimal character. This then implies that the representation is indeed the double lift of the appropriate character of $\mathrm{O}(0,s)$, and in the case of $\pi_{\mathbb{1}}$, that it is the unique representation (with positive signs) of $\mathrm{O}(2m,r)$ having the Knapp representation π_s' as a summand in its restriction to the identity component $\mathrm{SO}_e(2m,r)$. Moreover, π_s' is then uniquely determined be its lowest K-type and infinitesimal character. In [13], Knapp conjectured that this should typically be the case for the representations π'_s, and he gives a conjectural method for determining the Langlands parameters, which Friedman [6] proves to yield the correct parameters.

Proposition 5.21. *Let m, r, s, $\pi_{\mathbb{1}}$ and $\pi_{\det}$ be as above.*

(1) *If $\frac{r+s}{2} - m \neq 1$ then $\pi_{\mathbb{1}}$ is the unique irreducible admissible representation of $\mathrm{O}(2m, r)$ with infinitesimal character ν_s and lowest K-type Λ_s.*
(2) *If $s = m$ then $\pi_{\det}$ is the unique irreducible admissible representation of $\mathrm{O}(2m, r)$ with infinitesimal character ν_s and lowest K-type Λ'_s.*

Proof. The spherical case $\frac{r+s}{2} - m = 0$ for $\pi_{\mathbb{1}}$ follows from the well-known fact that any spherical representation of $\mathrm{O}(p, q)$ is determined by its infinitesimal character. For $\pi_{\mathbb{1}}$ and the case $\frac{r+s}{2} - m \geq 2$ or $\pi_{\det}$ with $m = s$, one computes the Vogan parameter λ_a associated to Λ_s (or Λ'_s) (see §5.3 of [33]), which essentially determines the Levi factor MA and the discrete part λ_d, μ of the parameters. Then one checks that there is only one parameter ν giving the given infinitesimal character, and the root system Ψ is then determined by the given unique lowest K-type. We give some details of the argument for $\pi_{\mathbb{1}}$ and the case $r \leq 2m + s$ with $\frac{r-s}{2} + m$ and $r = 2n$ even, and leave the remaining cases to the reader.

We have

$$\Lambda_s = \left(\frac{r+s}{2} - m, \frac{r+s}{2} - m, \dots, \frac{r+s}{2} - m; 0, \dots, 0\right), \tag{5.22}$$

so if ρ_c is one half the sum of the positive compact roots, then

$$\begin{aligned}\Lambda_s + 2\rho_c = \Bigg(&\underbrace{\frac{r+s}{2} + m - 2, \frac{r+s}{2} + m - 4, \dots, r+2, r}_{\frac{m}{2} - \frac{r-s}{4}}, \underbrace{r-2, r-4, \dots, \frac{r+s}{2} - m}_{\frac{m}{2} + \frac{r-s}{4}};\\ &\underbrace{r-2, r-4, \dots, \frac{r+s}{2} - m}_{\frac{m}{2} + \frac{r-s}{4}}, \frac{r+s}{2} - m - 2, \dots, 2, 0\Bigg).\end{aligned} \tag{5.23}$$

We may choose ρ (so that $\Lambda_s + 2\rho_c$ is dominant) to be

$$\begin{aligned}\rho = \Bigg(&\underbrace{\frac{r}{2} + m - 1, \frac{r}{2} + m - 2, \dots, \frac{3r-s}{4} + \frac{m}{2}}_{\frac{m}{2} - \frac{r-s}{4}}, \underbrace{\frac{3r-2}{4} + \frac{m}{2} - 1, \frac{3r-2}{4} + \frac{m}{2} - 3, \dots, \frac{r+2}{4} - \frac{m}{2} + 1}_{\frac{m}{2} + \frac{r-s}{4}};\\ &\underbrace{\frac{3r-2}{4} + \frac{m}{2} - 2, \frac{3r-2}{4} + \frac{m}{2} - 4, \dots, \frac{r+2}{4} - \frac{m}{2}}_{\frac{m}{2} + \frac{r-s}{4}}, \frac{r+2}{4} - \frac{m}{2} - 1, \frac{r+2}{4} - \frac{m}{2} - 2, \dots, 1, 0\Bigg),\end{aligned} \tag{5.24}$$

so that

$$(5.25)\quad \Lambda_s + 2\rho_c - \rho = \Big(\frac{s}{2}-1, \frac{s}{2}-2, \dots, \frac{r+s}{4}-\frac{m}{2}, \underbrace{\frac{r+s}{4}-\frac{m}{2}-1, \dots, \frac{r+s}{4}-\frac{m}{2}-1}_{\frac{m}{2}+\frac{r-s}{4}};$$
$$\underbrace{\frac{r+s}{4}-\frac{m}{2}, \dots, \frac{r+s}{4}-\frac{m}{2}}_{\frac{m}{2}+\frac{r-s}{4}}, \frac{r+s}{4}-\frac{m}{2}-1, \frac{r+s}{4}-\frac{m}{2}-2, \dots, 1, 0\Big).$$

Projection onto the dominant (w. r. t. ρ) Weyl chamber yields

$$(5.26)\quad \lambda_a = \Big(\frac{s}{2}-1, \frac{s}{2}-2, \dots, \frac{r+s}{4}-\frac{m}{2}, \underbrace{\frac{r+s}{4}-\frac{m}{2}-\frac{1}{2}, \dots, \frac{r+s}{4}-\frac{m}{2}-\frac{1}{2}}_{\frac{m}{2}+\frac{r-s}{4}};$$
$$\underbrace{\frac{r+s}{4}-\frac{m}{2}-\frac{1}{2}, \dots, \frac{r+s}{4}-\frac{m}{2}-\frac{1}{2}}_{\frac{m}{2}+\frac{r-s}{4}}, \frac{r+s}{4}-\frac{m}{2}-1, \frac{r+s}{4}-\frac{m}{2}-2, \dots, 1, 0\Big).$$

All entries of λ_a that occur only once will be entries of the Harish-Chandra parameter λ_d, and since $\frac{r+s}{4}-\frac{m}{2}-\frac{1}{2} > 0$, the Levi factor MA will be of the form

$$(5.27)\qquad MA \cong \mathrm{O}(2m-2t, r-2t) \times \mathrm{GL}(2,\mathbb{R})^t$$

for some $0 \le t \le \frac{m}{2}+\frac{r-s}{4}$.

Recall that the infinitesimal character (up to Weyl group action) is

$$(5.28)\quad \nu_s = \Big(\frac{s}{2}-1, \frac{s}{2}-2, \dots, \frac{r+s}{4}-\frac{m}{2}, \frac{r+s}{4}-\frac{m}{2}-1, \dots, \frac{s}{2}-m;$$
$$\frac{r}{2}-1, \frac{r}{2}-2, \dots, \frac{r+s}{4}-\frac{m}{2}, \frac{r+s}{4}-\frac{m}{2}-1, \dots, 1, 0\Big).$$

Since ν_s does not contain $\frac{r+s}{4}-\frac{m}{2}-\frac{1}{2}$ as an entry, it can not be an entry in λ_d, so we must have $t = \frac{m}{2}+\frac{r-s}{4}$ and

$$(5.29)\qquad \lambda_d = \left(\frac{s}{2}-1, \frac{s}{2}-2, \dots, \frac{r+s}{4}-\frac{m}{2}; \frac{r+s}{4}-\frac{m}{2}-1, \frac{r+s}{4}-\frac{m}{2}-2, \dots, 1, 0\right).$$

A relative limit of discrete series of $\mathrm{GL}(2,\mathbb{R})^t$ parametrized by a pair (μ,ν) with $\mu = (\mu_1,\dots,\mu_t) \in \left(\mathbb{Z}_+\right)^t$ and $\nu = (n_1,\dots,n_t) \in \mathbb{C}^t$ has infinitesimal character $\frac{1}{2}(\mu_1+n_1,\dots,\mu_t+n_t,-\mu_1+n_1,\dots,-\mu_1+n_t)$. In order to account for the remaining entries $(\frac{r}{2}-1, \frac{r}{2}-2, \dots, 1, 0, -1, \dots, \frac{s}{2}-m)$ of ν_s, we must have $\mu = (\frac{r+s}{2}-m-1, \frac{r+s}{2}-m-1, \dots, \frac{r+s}{2}-m-1)$ and $\nu = (\frac{r+s}{2}+m-1, \frac{r+s}{2}+m-3, \dots, 3, 1)$. (Note that changing the sign on an entry of ν does not change the equivalence class of the corresponding representation of $\mathrm{O}(2m,r)$.) $\square$

To illustrate how Proposition 5.21 fails in the other cases, we look at three examples; the first example deals with the ambiguities in Langlands parameters for representations with lowest K-type Λ_s when $\frac{r+s}{2}-m=1$, the other two look at representations which have the same infinitesimal character and lowest K-type as $\pi_{\det}$.

Example 5.30. Let $m = 3$, $s = 2$, and $r = 6$, and consider $\pi_{\mathbb{1}}$, a representation of $\mathrm{O}(6,6)$ with lowest K-type $\Lambda_2 = (1,1,1;0,0,0)$ (a fine K-type) and infinitesimal character $\nu_2 = (2,2,1,1,0,0)$. The Langlands parameters are given by $MA \cong \mathrm{O}(2,2) \times \mathrm{GL}(2,\mathbb{R})^2$, $\lambda_d = (0;0)$, $\Psi = \{e_1 \pm f_1\}$, $\mu = (0,0)$, and $\nu = (4,2)$. Notice that the associated Vogan parameter $\lambda_a = (0,0,0;0,0,0)$, so $L = \mathrm{O}(6,6)$, $\rho(\mathfrak{u} \cap \mathfrak{p})$ and $\rho(\mathfrak{u} \cap \mathfrak{k})$ are zero as well, and any representation containing Λ_2 as a lowest K-type must be a constituent of a principal series representation. The full principal series will have a second lowest K-type $(0,0,0;1,1,1)$. The particular principal series which has $\pi_{\mathbb{1}}$ as a constituent is obtained by inducing the character given by $\epsilon = (1,1,1,-1,-1,-1)$ and $\kappa = (0,1,2,0,1,2)$ on $MAN = \mathrm{GL}(1,\mathbb{R})^6 \cdot N$ to $\mathrm{O}(6,6)$ (this is the character mapping $(r_1,r_2,r_3,r_4,r_5,r_6) \in (\mathbb{R}^\times)^6$ to $|r_2||r_3|^2 sign(r_4) sign(r_5)|r_5| sign(r_6)|r_6|^2$). The constituent containing the other lowest K-type has Langlands parameters which differ from those of $\pi_{\mathbb{1}}$ only by the choice of positive roots Ψ (the corresponding limit of discrete series is antiholomorphic). We can get other representations with these lowest K-types and the same infinitesimal character by matching the entries of κ with the entries of ϵ in different ways. In this way (and eliminating duplications by ensuring that the nonparity condition F-2 of [34] is satisfied) we get precisely three pairwise inequivalent irreducible admissible representations of $\mathrm{O}(6,6)$ with unique lowest K-type Λ_2 and infinitesimal character ν_2, namely $\pi_{\mathbb{1}}$, $\pi(\lambda_d, \Psi, 0, 0, \epsilon', \kappa')$ with $MA \cong \mathrm{O}(2,2) \times \mathrm{GL}(1,\mathbb{R})^4$, $\epsilon' = (1,1,-1,-1)$ and $\kappa' = (2,2,1,1)$, and $\pi(\lambda_d, \Psi, 0, 0, \epsilon', \kappa'')$ with MA as for the previous representation, and $\kappa'' = (1,1,2,2)$.

Once we assume that our Levi factor is $MA \cong \mathrm{O}(2,2) \times \mathrm{GL}(2,\mathbb{R})^2$, the remaining parameters are uniquely determined, and this is the only type of ambiguity which occurs for the Langlands parameters of representations which have the same infinitesimal character and lowest K-type as $\pi_{\mathbb{1}}$.

Example 5.31. An ambiguity similar to the one dealt with in Example 5.30 occurs with $\pi_{\det}$ when Λ'_s is small, i. e., of the form $(2,\dots,2,1,\dots,1;0,\dots,0)$. Let $s = 2$, $m = 3$, and $r = 6$ as before. Then $\Lambda'_2 = (2,2,1;0,0,0)$, $\nu_2 = (2,2,1,1,0,0)$, and the Langlands parameters of $\pi_{\det}$ are given by $MA \cong \mathrm{O}(2,2) \times \mathrm{GL}(2,\mathbb{R})^2$, $\lambda_d = (1;0)$, $\mu = (1,0)$, and $\nu = (1,4)$. The Vogan parameter is given by $\lambda_a = (1, \frac{1}{2}, 0; \frac{1}{2}, 0, 0)$, so there are two more irreducible admissible representations with this lowest K-type and infinitesimal character, namely with $MA \cong \mathrm{O}(2,2) \times \mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R})^2$, λ_d as above, $\mu = (1)$, $\nu = (3)$, $\epsilon = (1,-1)$, and $\kappa = (2,0)$ or $(0,2)$.

Example 5.32. If Λ'_s is not small as in Example 5.31 then the Levi factor part of the Langlands parameters is uniquely determined by the lowest K-type and its uniqueness. We get a different kind of ambiguity from the fact that if $s \neq m$ then MA contains $\mathrm{GL}(2,\mathbb{R})$ factors with two different discrete parameters attached, and there may be more than one way to match continuous parameters and get the same infinitesimal character. For instance, let $s = 4$, $m = 6$, and $r = 14$. Then $\Lambda'_4 = (4,4,4,4,3,3;0,0,0,0,0,0,0)$, $\nu_2 = (6,5,4,4,3,3,2,2,1,1,1,1,0,0)$, and $\pi_{\det}$ has Langlands parameters $MA = \mathrm{O}(2,4) \times \mathrm{GL}(2,\mathbb{R})^5$, $\lambda_d = (2;1,0)$, $\mu = (3,3,3,2,2)$ and $\nu = (5,3,1,10,8)$. Continuous parameters $(9,5,5,8,2)$ or $(7,5,3,10,4)$ give two more inequivalent representations with the same lowest K-type and infinitesimal character.

The next result shows how to introduce additional hypotheses to circumvent the ambiguities of the previous examples.

Theorem 5.33. *Let m, r, s, $\pi_{\mathbb{1}}$ and $\pi_{\det}$ be as above. Recall the maximal primitive ideal $\mathrm{J}_{\max}(\nu_s)$ with infinitesimal character ν_s (Section 2.3).*

(1) *$\pi_{\mathbb{1}}$ is the unique irreducible admissible representation of $\mathrm{O}(2m,r)$ with annihilator $\mathrm{J}_{\max}(\nu_s)$ and lowest K-type Λ_s.*
(2) *$\pi_{\det}$ is the unique irreducible admissible representation of $\mathrm{O}(2m,r)$ with annihilator $\mathrm{J}_{\max}(\nu_s)$ and lowest K-type Λ'_s.*

Sketch. We begin by recalling the τ-invariant of an irreducible $\mathrm{U}(\mathfrak{g})$ module π with regular integral infinitesimal character. For the purposes of this paragraph, $\mathfrak{g}$ may be taken to be an arbitrary complex semisimple Lie algebra. Fix a positive system of roots Δ^+ for a Cartan $\mathfrak{h}$ in $\mathfrak{g}$. Let λ denote a dominant (with respect to Δ^+) representative of the infinitesimal character of π. Fix a simple root $\alpha \in \Delta^+$ and let $\lambda_\alpha = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. Then λ_α is integral and dominant (with respect to Δ^+); moreover, λ and λ_α differ by a weight of a finite dimensional representation of $\mathfrak{g}$. We may thus consider the translation functor ψ_α from λ to λ_α. The τ-invariant of π, denoted $\tau(\pi)$, is the set of simple α for which $\psi_\alpha(\pi) = 0$.

Return to our setting and let π be an irreducible admissible representation for $\mathrm{O}(2m, r)$. We now extract a consequence of the classification of primitive ideals in $\mathfrak{g} = \mathfrak{so}(2m + r, \mathbb{C})$. Assume π has infinitesimal character ν_s and make the standard choices (of Cartan, positive roots Δ^+, etc.) so that the formulas (2.7) and (2.9) give the dominant representative of the infinitesimal character of π. Since we are assuming the parity of r and s match, ν_s is integral. The translation principle dictates the following: there is a regular infinitesimal character with dominant representative ν_s^{reg} so that ν_s^{reg} and ν differ by the weight of a finite-dimensional representation of $\mathfrak{g}$; and there is a representation π^{reg} with infinitesimal character ν_s^{reg} so that if T denotes the translation functor from ν_s^{reg} to ν, then

$$T(\pi^{\mathrm{reg}}) = \pi.$$

Now let S_s denote the set of simple roots $\alpha \in \Delta^+$ for which ν_s is *not* singular, i.e. those simple roots so that $\langle \nu_s, \alpha \rangle \neq 0$. We claim that

$$\mathrm{J}_{\max}(\nu_s) \text{ annihilates } \pi \text{ iff } S_s = \tau(\pi^{\mathrm{reg}}).$$

This follows from the classification of primitive ideals in $\mathrm{U}(\mathfrak{g})$. In more detail, the paper [29] gives the explicit tableau parameters of the primitive ideal $\mathrm{J}_{\max}(\nu_s^{\mathrm{reg}})$, and from there it is simple to extract the τ-invariant statement.

Now we sketch how to use the τ-invariant criterion to rule out the ambiguities highlighted in Examples 5.30–5.32. Our task is to take a representation π (different from $\pi_{\mathbf{1}}$ and $\pi_{\det}$) given by one of the Langlands parameters in those examples, compute the τ-invariant of π^{reg}, and show that it differs from S_s. Here is a sketch of how to do that. The papers [32] and [34] explain how to produce the Langlands parameters of π^{reg} and then to compute the τ-invariant of π^{reg}. (Theorem 4.12 in [32] is especially relevant.) Then one can see directly that the τ-invariant of π^{reg} is strictly *smaller* than S_s. For instance, suppose we encounter the ambiguity of the sort treated in Examples 5.30 and 5.31. In these cases the Levi factor contains more $\mathrm{GL}(1,\mathbb{R})$ factors than the Levi factor for π_1. The real roots that arise from these additional $\mathrm{GL}(1)$ factors cannot satisfy the parity condition. This corresponds to saying that the real roots are *not* in the τ-invariant of π^{reg}, and thus the τ-invariant is smaller than possible and, in particular, smaller than S_s. The final kind of ambiguity treated in Example 5.32 is slightly more subtle. One must verify that the alternative matchings of continuous parameters always lead to a smaller τ-invariant for π^{reg}. Given [32, Theorem 4.12], this is a rather complicated combinatorial check. We omit the details. □

As a corollary, we immediately obtain the Langlands parameters of the double lifts $\theta(\mathbf{1}_s)$ and $\theta(\det_s)$.

Corollary 5.34. *Let m, r, s, $\pi_{\mathbb{1}}$ and $\pi_{\det}$ be as above. Then*

$$\pi_{\mathbf{1}} = \theta^2(\mathbf{1}_s) \text{ and } \pi_{\det} = \theta^2(\det_s).$$

Proof. Since the double lifts $\theta(\mathbf{1}_s)$ and $\theta(\det_s)$ satisfy the hypothesis of the theorem (with the annihilator hypothesis explained in the proof of Proposition 4.3), the corollary follows. □

The proof of Proposition 4.1 is also now a simple corollary: if there were more than one representation of the kind described in the proposition, there would be more than one of the kind described in Theorem 5.33. This contradiction completes the proof. □

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