SPECIAL UNIPOTENT REPRESENTATIONS AND THE HOWE CORRESPONDENCE

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Abstract. We use the Howe correspondence to establish new unitarity results for special unipotent representations of certain classical real Lie groups.

Let $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ be a reductive dual pair in $\mathrm{Sp}(2N,\mathbb{R})$. Let $\widetilde{G}_{\mathbb{R}}$ and $\widetilde{G}'_{\mathbb{R}}$ be the preimages of $G_{\mathbb{R}}$ and $G'_{\mathbb{R}}$ in $\mathrm{Mp} = \mathrm{Mp}(2N,\mathbb{R})$, the nontrivial double cover of Sp. Write $\mathrm{Irr}(\widetilde{G}_{\mathbb{R}})$ for the set of equivalence classes of irreducible admissible representations for $G_{\mathbb{R}}$, and likewise for $\mathrm{Irr}(\widetilde{G}'_{\mathbb{R}})$. Two representations $\pi \in \mathrm{Irr}(\widetilde{G}_{\mathbb{R}})$ and $\pi' \in \mathrm{Irr}(\widetilde{G}'_{\mathbb{R}})$ are said to correspond if $\pi \otimes \pi'$ is a quotient of a fixed oscillator representation for Mp; in this case π and π' are said to occur in the correspondence. Howe proved that the map $\pi \mapsto \pi'$ is well-defined and bijective when restricted to those representations which occur [H2]. Hence we obtain a map

$$\theta : \mathrm{Irr}(\widetilde{G}_{\mathbb{R}}) \longrightarrow \mathrm{Irr}(\widetilde{G}'_{\mathbb{R}}) \cup \{0\},$$

where $\theta(\pi) = 0$ if π does not occur in the correspondence. (This map depends on a choice an oscillator representations for Mp.) In most cases there exists genuine characters χ and χ' of $\widetilde{G}_{\mathbb{R}}$ and $\widetilde{G}'_{\mathbb{R}}$ both of which have infinitesimal character zero. Given $\pi \in \mathrm{Irr}(G_{\mathbb{R}})$, we will often be sloppy and write $\theta(\pi)$ for $\theta(\pi \otimes \chi) \otimes \chi'$ which is a nongenuine representation of $\widetilde{G}'_{\mathbb{R}}$ and which we thus view as an element of $\mathrm{Irr}(G'_{\mathbb{R}})$. This bit of imprecision is customary and we henceforth ignore it in the introduction.

It has long been observed that interesting small unitary representations of $G_{\mathbb{R}}$ correspond to interesting small unitary representations of $G'_{\mathbb{R}}$. The idea goes back to Howe himself and has been exploited by many authors. (A very incomplete list of some highlights includes [H1], [Li1], [Li2], [Pr1], and [He]; see also the references given there.) The purpose of this note is to sharpen these ideas in the context of Arthur's special unipotent representations.

For orientation, we recall the definition of a special unipotent representation of a linear real reductive group $G_{\mathbb{R}}$. We need some general notation first. Let $\mathfrak{g}_{\mathbb{R}}$ denote the Lie algebra of $G_{\mathbb{R}}$, let $\mathfrak{g}$ denote its complexification, and write $\mathfrak{g}^\vee$ for the dual Lie algebra. Fix a nilpotent adjoint orbit $\mathcal{O}^\vee$ in $\mathfrak{g}^\vee$, and choose $f^\vee \in \mathcal{O}^\vee$. According to the Jacobsen-Morozov Theorem, there exists a Lie algebra homomorphism ϕ from $\mathfrak{sl}(2,\mathbb{C})$ to $\mathcal{O}^\vee$ such that $\phi\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = f^\vee$. Define $h^\vee = \phi\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, and let $\mathfrak{h}^\vee$ denote a Cartan containing $h^\vee$. Then $\mathfrak{h}^\vee$ is canonically isomorphic to the linear dual $\mathfrak{h}^*$ of a Cartan $\mathfrak{h}$ of $\mathfrak{g}$. Finally consider

$$\chi(\mathcal{O}^\vee) = \frac{1}{2}h^\vee \in \mathfrak{h}^\vee \simeq \mathfrak{h}^*.$$

There were a variety of choices made in the definition of $\chi(\mathcal{O}^\vee)$. Different choices amount at most to modifying $\chi(\mathcal{O}^\vee) \in \mathfrak{h}^*$ by an element of $W = W(\mathfrak{h},\mathfrak{g})$. In other words,

$$\chi(\mathcal{O}^\vee) \text{ is a well-defined element of } \mathfrak{h}^*/W,$$

so defines an infinitesimal character for $\mathfrak{g}$. The following definition is perhaps the simplest possible, but masks most of the deeper ideas that are behind making it.

Definition 0.1 (Arthur, cf. [BV1]). An irreducible representation π of a linear reductive group $G_{\mathbb{R}}$ is called *(integral) special unipotent* if: (1) its infinitesimal character is integral and of the form $\chi(\mathcal{O}^\vee)$; and (2) the GK-dimension of π is minimal among all representations with the same infinitesimal character as π. In this case, π is said to be attached to $\mathcal{O}^\vee$. (The condition that $\chi(\mathcal{O}^\vee)$ be integral is equivalent to requiring that $\mathcal{O}^\vee$ is even.)

The basic example is the trivial representation. Its infinitesimal character is ρ which is $\frac{1}{2}h^\vee$ for the principal orbit $\mathcal{O}^\vee$, and of course it is the smallest representation with such infinitesimal character. At the other extreme are representations with infinitesimal character zero which is $\frac{1}{2}\mathfrak{h}^\vee$ for the zero orbit $\mathcal{O}^\vee$. Such representations exist if and only if G is quasisplit and in that case they all have the same (maximal) GK-dimension. In other words, any representation with infinitesimal character zero is special unipotent.

The definition of special unipotent is entirely algebraic. Yet Arthur's conjectures predict that such representations should be local components of global automorphic forms; in particular, they should be unitary.

Conjecture 0.2 (Arthur). *If π is special unipotent, then π is unitary.*

The best partial progress on this conjecture is due to Barbasch for classical groups. (For instance, [B] establishes the complex classical case.) For real groups of type A the conjecture follows from work of Barbasch, Speh, and Vogan. For other real groups (apart from those of small real rank), the conjecture is still open. We note that both of our examples given above — the trivial representation and representations with infinitesimal character zero — are unitary. (The latter are all limits of discrete series [V1].)

In general, $\chi(\mathcal{O}^\vee)$ is singular — it is regular only if $\mathcal{O}^\vee$ is principal — so condition (1) implies that π cannot be too small. On the other hand condition (2) says that π cannot be too large. Thus special unipotent representation exist on the interface of these restrictions. The key point is that the Howe correspondence preserves this interface in a sense which we now explain.

Suppose π is special unipotent and assume that $\theta(\pi) \neq 0$. Przebinda [Pr2] computed the infinitesimal character of $\theta(\pi)$ in terms of the infinitesimal character, say λ, of π, and it is easy to verify that if $\lambda = \chi(\mathcal{O}^\vee)$, then there typically exists a nilpotent orbit $\mathcal{O}_1^\vee$ for $(\mathfrak{g}')^\vee$ such that the infinitesimal character of $\theta(\pi)$ is $\chi(\mathcal{O}_1^\vee)$. On the other hand Przebinda [Pr3] computed an upper bound on the size of $\theta(\pi)$ as follows. Let $\mathcal{O}_{\min}$ denote the minimal coadjoint orbit in $\mathfrak{sp}(2N,\mathbb{C})^*$; in other words $\overline{\mathcal{O}_{\min}}$ is the associated variety of the annihilator of the oscillator used to define the correspondence. Let μ denote the projection from $\overline{\mathcal{O}_{\min}}$ to $\mathfrak{g}^*$ obtained by first considering the projection $\mathfrak{sp}(2N,\mathbb{C})^*$ to $\mathfrak{g}^*$ (dual to inclusion) and then by restricting the domain to $\overline{\mathcal{O}^{\min}}$. Likewise write μ' for the projection from $\overline{\mathcal{O}_{\min}}$ to $(\mathfrak{g}')^*$. Let $\overline{\mathcal{O}}$ denote the associated variety of the annihilator of π. [DKP1] proved that $\mu'(\mu^{-1}(\overline{\mathcal{O}_{\min}}))$ is the closure of a single coadjoint orbit for $\mathfrak{g}'$; we write $\theta(\mathcal{O})$ for this orbit. The result of Przebinda giving an upper bound on the size of $\theta(\pi)$ may now be written as

$$\mathrm{AV}(\mathrm{Ann}(\theta(\pi))) \subset \overline{\theta(\mathrm{AV}(\mathrm{Ann}(\pi)))}; \tag{1}$$

and, in particular,

$$\text{the GK dimension of } \pi \leq \text{ the dimension of } \theta(\mathcal{O}).$$

Since the infinitesimal character of a special unipotent representation is dictated by nilpotent orbits in $\mathfrak{g}^\vee$, and the size estimate is dictated by nilpotent orbits in $\mathfrak{g}$, we need a mechanism to pass between the two kinds of orbits. This is provided for by Spaltenstein's duality map d (Section 1.6). A short computation (Section 1.9) then gives the following result.

Theorem 0.3. *Consider a dual pair of the form $(G_\mathbb{R}, G'_\mathbb{R}) = (\mathrm{Sp}(p,q), \mathrm{O}^*(2m))$ or $(G_\mathbb{R}, G'_\mathbb{R}) = (\mathrm{O}^*(2m), \mathrm{Sp}(p,q))$. Suppose $\pi \in \mathrm{Irr}(\widetilde{G}_\mathbb{R})$ is special unipotent attached to $\mathcal{O}^\vee$. Then $\theta(\pi)$ is either zero or special unipotent attached to $d(\theta(d(\mathcal{O}^\vee)))$.*

A similar result is valid for any dual pair, but some additional technical hypothesis are needed. Some of the subtleties in the general case are sketched in Section 4.

We wish to pursue the following idea. In some cases (the "stable range"), Li [Li1] proved that if π is nonzero and unitary then $\theta(\pi)$ is also nonzero and unitary. Thus if we can establish Conjecture 0.2 for a special unipotent representation π, then Theorem 0.3 implies Conjecture 0.2 for $\theta(\pi)$. The trivial representation is the simplest special unipotent representation. Beginning with it, and taking iterated stable-range theta lifts, we thus obtain a large collection of unitary special unipotent representations. The issue of exhaustion then remains: can one obtain all special unipotent representations as iterated lifts from the trivial representation? The latter question can be approached using the counting techniques of [BV1] and [Mc1]. For $\mathrm{Sp}(p,q)$ and $\mathrm{O}^*(2m)$, we arrive at the following result.

Theorem 0.4. *Let $G_\mathbb{R} = \mathrm{Sp}(p,q)$ and suppose $\mathcal{O}^\vee$ is an even nilpotent orbit for $\mathfrak{so}(2p+2q+1, \mathbb{C})$ with the following condition: in the partition classification of $\mathcal{O}^\vee$ according to Jordan form, assume that each part p of the partition corresponding to $\mathcal{O}^\vee$ is greater than the sum of all other parts less than or equal to p. Suppose π is a special unipotent representation attached to $\mathcal{O}^\vee$ (Definition 0.1). Then π is unitary.*

More precisely, π is obtained by a sequence of iterated theta lifts from the trivial representation as follows. There exists a sequence of dual pairs each of which is in the stable range

$$(\mathrm{O}^*(2m_1), \mathrm{Sp}(p_1,q_1)), (\mathrm{Sp}(p_1,q_1), \mathrm{O}^*(2m_2)), (\mathrm{O}^*(2m_2), \mathrm{Sp}(p_2,q_2)), \ldots, (\mathrm{O}^*(2m_k), \mathrm{Sp}(p,q))$$

so that if the corresponding theta lifts are denoted by

$$\mathrm{Irr}(\mathrm{O}^*(2m_1)) \xrightarrow{\theta_1} \mathrm{Irr}(\mathrm{Sp}(p_1,q_1)) \xrightarrow{\theta_2} \mathrm{Irr}(\mathrm{O}^*(2m_2)) \xrightarrow{\theta_3} \cdots \xrightarrow{\theta_{2k-1}} \mathrm{Irr}(\mathrm{Sp}(p,q)),$$

then

$$\pi = [\theta_{2k-1} \circ \theta_{2k} \circ \cdots \circ \theta_1](\mathbb{1}).$$

The identical statements hold (with obvious modifications) for $G_\mathbb{R} = \mathrm{O}^*(2n)$.

The theorem verifies a large part of Conjecture 0.2 for $\mathrm{Sp}(p,q)$ and $\mathrm{O}^*(2n)$ and is proved after Theorem 3.7 below. The condition on the partition corresponding to $\mathcal{O}^\vee$ is a vestige of the fact that each pair used for the iterated lift must be in the stable range so that the results of [Li1] apply. This condition may perhaps be omitted completely using the results of [He] (in which case the full Conjecture 0.2 would follow for the groups in question). Results of an identical nature for $\mathrm{U}(p,q)$, $\mathrm{GL}(n,\mathbb{C})$, and $\mathrm{GL}(n,\mathbb{R})$ also follow. But as remarked above, special unipotent representations for these groups are already well-understood and we get no new unitarity results. Nonetheless the case of $\mathrm{U}(p,q)$ does exhibit some intriguing features; we give complete details in Section 2.

The same kinds of ideas work for arbitrary dual pairs, but there are significant complications. We already noted some additional hypothesis are needed for Theorem 0.3. More importantly, the count required to prove exhaustion fails in a mildly complicated way for the pairs $(\mathrm{O}(p,q), \mathrm{Sp}(2n,\mathbb{R}))$ (owing to the disconnectedness of the orthogonal group). Some examples are given in Section 4.

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1. Background: nilpotent orbits and the theta correspondence

1.1. **General notation.** Let $G_\mathbb{R}$ be a real reductive group with maximal compact subgroup $K_\mathbb{R}$ corresponding to a Cartan involution τ. We write $\mathfrak{g}_\mathbb{R}$ and $\mathfrak{k}_\mathbb{R}$ for the corresponding Lie algebras, $\mathfrak{g}$ and $\mathfrak{k}$ for their complexifications, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the corresponding Cartan involution. We often implicitly identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of a fixed invariant form. In the setting of a reductive dual pair $(G_\mathbb{R}, G'_\mathbb{R})$ in $\mathrm{Sp}(2N,\mathbb{R})$, we adopt the analogous notation for $G'_\mathbb{R}$ but simply add a prime everywhere. For instance $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ is the complexified Cartan decomposition with respect to τ'. We further assume that τ and τ' are the restriction of a fixed Cartan involution for $\mathrm{Sp}(2N,\mathbb{R})$.

1.2. **Nilpotent orbits in complex classical Lie algebras.** Let $\mathfrak{g}$ denote a complex reductive Lie algebra and write G for a connected complex group with Lie algebra G. Let $\mathcal{N}(\mathfrak{g})$ denote the nilpotent cone in $\mathfrak{g}$. G acts on $\mathcal{N}(\mathfrak{g})$ with finitely many orbits. Using an invariant bilinear form, $\mathcal{N}(\mathfrak{g})$ identifies with $\mathcal{N}(\mathfrak{g}^*)$, the nilpotent cone in $\mathfrak{g}^*$, and the orbits of G on $\mathcal{N}(\mathfrak{g})$ and $\mathcal{N}(\mathfrak{g}^*)$ are also identified.

Recall the nilpotent orbits in $\mathfrak{sl}(n,\mathbb{C})$ are classified according to Jordan normal forms of elements, i.e. by partitions of n. We denote the set of all such partition by $\Pi(n)$ or $\Pi_A(n)$.

For $\mathfrak{sp}(2n,\mathbb{C})$, nilpotent orbits are again classified according to Jordan normal forms. The partitions that arise in this was have all odd parts occurring with even multiplicity. We write $\Pi_C(2n)$ for such partitions.

Nilpotent orbits in $\mathfrak{so}(2n+1,\mathbb{C})$ are classified by the set $\Pi_B(2n+1)$ of partitions of $2n+1$ in which even parts occur with even multiplicity.

Finally let $\Pi_D(2n)$ denote the set of partition of $2n$ in which even parts occur with even multiplicity. The adjoint orbits of the disconnected group $\mathrm{O}(2n,\mathbb{C})$ on $\mathfrak{so}(2n,\mathbb{C})$ are parametrized by $\Pi_D(2n)$. The parametrization of orbits of $\mathrm{SO}(2n,\mathbb{C})$ is slightly more elaborate: the partition in $\Pi_D(2n)$ has only even parts if and only if the $\mathrm{O}(2n,\mathbb{C})$ orbit splits into two $\mathrm{SO}(2n,\mathbb{C})$ orbits; but we shall have no occasion to study the latter orbits.

Notation 1.1. Given a complex semisimple Lie algebra $\mathfrak{g}$, we let $\mathbf{N}(\mathfrak{g})$ denote the set of G orbits on $\mathcal{N}(\mathfrak{g})$ with one exception. For applications to the theta correspondence, it is only the disconnected orthogonal group which arises. *Consequently we shall only consider nilpotent orbits of* $\mathrm{O}(n,\mathbb{C})$ *on* $\mathcal{N}(\mathfrak{so}(n,\mathbb{C}))$ *in this paper.* We deviate from our usual notation and write $\mathbf{N}(\mathfrak{g})$ for this set in this case. We use analogous notation for $\mathbf{N}(\mathfrak{g}^*)$.

1.3. **Associated varieties of primitive ideals.** Let $\mathfrak{U}(\mathfrak{g})$ denote the universal enveloping algebra of a complex reductive Lie algebra $\mathfrak{g}$ and let X denote a simple $\mathfrak{U}(\mathfrak{g})$ module. Let I_X denote the two sided ideal $\mathrm{Ann}_{\mathfrak{U}(\mathfrak{g})}(X)$. Consider the degree filtration on $\mathfrak{U}(\mathfrak{g})$. Passing to associated graded objects gives an ideal $\mathrm{gr} I_X$ in $S(\mathfrak{g})$. The support of $\mathrm{gr} I_X$ is a G-invariant subvariety of $\mathfrak{g}^*$ called the associated variety of the annihilator of X and denoted

$\mathrm{AV}(\mathrm{Ann}(X))$. Since X has finite length, it is annihilated by an ideal of finite codimension in the center $Z(\mathfrak{g})$ of $\mathfrak{U}(\mathfrak{g})$. Since the center cuts out $\mathcal{N}(\mathfrak{g}^*)$, and since $\mathrm{AV}(\mathrm{Ann}(X))$ is invariant under the adjoint action, $\mathrm{AV}(\mathrm{Ann}(X))$ is a union of nilpotent orbits. In fact, if X is irreducible, $\mathrm{AV}(\mathrm{Ann}(X))$ is the closure of a single orbit ([BB]).

1.4. **Associated varieties of Harish-Chandra modules.** We recall the main construction of [V4]. Fix $G_{\mathbb{R}}$, and let X be an irreducible $(\mathfrak{g}, K)$ module. Fix a K-stable good filtration of X, and consider the $S(\mathfrak{g})$ module obtained by passing to the associated graded object $\mathrm{gr}(X)$. By identifying $(\mathfrak{g}/\mathfrak{k})^*$ with $\mathfrak{p}$ (and noting the K-invariance of the filtration), we can consider the support of $\mathrm{gr}(X)$ as a subvariety of $\mathfrak{p}$. This subvariety is called the associated variety of X and is denoted $\mathrm{AV}(X)$. It is a (finite) union of closures of elements of $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{p})$ where $\mathrm{AV}(\mathrm{Ann}(X)) = \overline{\mathcal{O}}$. Each such component is a nilpotent K orbit on $\mathfrak{p}$.

1.5. **Operations on nilpotent orbits in complex classical Lie algebras: adding a column.** Recall Przebinda's upper bound in Equation (1). The paper [DKP1] computes the upper bound explicitly in terms of partitions. In this section we recall a particularly simple case of that computation adequate for our purposes.

We now describe an operation on nilpotent orbits in terms of the partition classification given in Section 1.2. We first treat Type A. Fix $n \leq m$ and define a map

$$\theta \;:\; \Pi(n) \longrightarrow \Pi(m),$$

defined by augmenting the $m-n$ largest parts of a given partition by one. More precisely, given a partition λ,

$$n = n_1 + n_2 + \cdots + n_k$$

stretch it to have length n by padding it with zero entries $n_{k+1} = n_{k+2} = \cdots = n_m = 0$, and then define $\theta(\lambda)$ to be the partition

$$n = (n_1 + 1) + (n_2 + 1) + \cdots + (n_{m-n} + 1) + n_{m-n+1} + \cdots + n_m,$$

removing any terminal zeros as necessary.

Outside of Type A, the situation is necessarily more complicated. For instance, fix X of Type B, C, or D. Then it is easy to see that there exist n, m, and $\lambda \in \Pi_X(n)$ so that $\theta(\lambda) \notin \Pi_Y(m)$ for Y equal to B, C, or D. We sweep this under the rug by incorporating it into our hypotheses.

Proposition 1.2. *Let $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ denote an irreducible reductive dual pair and consider the corresponding complex Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$. Fix an orbit $\mathcal{O} = \mathcal{O}_\lambda$ in $\mathbf{N}(\mathfrak{g})$ (Notation 1.1) parametrized by a partition λ. Suppose the size of $\mathfrak{g}'$ is arranged so that the column-adding operation $\theta(\lambda)$ is a partition of the appropriate type to parametrize an orbit $\mathcal{O}'_{\theta(\lambda)}$ in $\mathbf{N}(\mathfrak{g}')$. Recall the orbit $\theta(\mathcal{O}_\lambda)$ defined just before Equation (1) in the introduction. Then*

$$\theta(\mathcal{O}_\lambda) = \mathcal{O}'_{\theta(\lambda)}.$$

Remark 1.3. Notice that outside of Type A, we have restricted the size of $\mathfrak{g}'$ so that the column-adding operation defines a partition of the appropriate type. [DKP1] computes $\theta(\mathcal{O}_\lambda)$ without this restriction, but we do not need the more general computation here.

Example 1.4. Fix $n \leq m$ and consider

$$(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{sp}(2n, \mathbb{C}), \mathfrak{so}(2m, \mathbb{C})) \text{ or } (\mathfrak{so}(2n, \mathbb{C}), \mathfrak{sp}(2m, \mathbb{C})).$$

Suppose λ is a partition in which all parts occur with even multiplicity. Then λ always defines an orbit for $\mathfrak{g}$. Moreover, $\theta(\lambda)$ always defines an orbit for $\mathfrak{g}'$. A "dual" fact is the subject of the first assertion in Proposition 1.10 below.

As as a consequence of Przebinda's result (Equation (1)), we obtain the following computable upper bound on the size of $\theta(\pi)$.

Corollary 1.5 ([Pr1], [DKP1]). *Retain the notation and hypotheses of Proposition 1.2. Suppose π is an irreducible representation of $\widetilde{G}_{\mathbb{R}}$ with $\mathrm{AV}(\mathrm{Ann}(\pi)) = \overline{\mathcal{O}_\lambda}$. Then*

$$\mathrm{AV}(\mathrm{Ann}(\theta(\pi))) \subset \overline{\mathcal{O}'_{\theta(\lambda)}}.$$

1.6. **Duality of orbits.** Let $\mathfrak{g}$ denote a complex simple Lie algebra and $\mathfrak{g}^\vee$ for its complex dual. Recall Notation 1.1. Spaltenstein defined a map

$$d \;:\; \mathbf{N}(\mathfrak{g}) \longrightarrow \mathbf{N}(\mathfrak{g}^\vee)$$

with many remarkable properties. For instance the image of d consists exactly of the special orbits in $\mathbf{N}(\mathfrak{g}^\vee)$ and d^2 is the identity when restricted to special orbits in $\mathbf{N}(\mathfrak{g})$. Here is the another property that is especially important for us.

Theorem 1.6 (Barbasch-Vogan). *Let $G_{\mathbb{R}}$ be a linear reductive group. Suppose π is a special unipotent representations of $G_{\mathbb{R}}$ attached to $\mathcal{O}^\vee$. Then*

$$\mathrm{AV}(\mathrm{Ann}(\pi)) = \overline{d(\mathcal{O}^\vee)}.$$

Proof. This is proved in Section A.3 of [BV1]. □

Corollary 1.7. *Let $G_{\mathbb{R}}$ be a linear reductive group and fix an even nilpotent orbit $\mathcal{O}^\vee$ in $\mathfrak{g}^\vee$. Suppose π is a representation of $G_{\mathbb{R}}$ with infinitesimal character $\chi(\mathcal{O}^\vee)$ such that*

$$\mathrm{AV}(\mathrm{Ann}(\pi)) \subset \overline{d(\mathcal{O}^\vee)}.$$

Then the inclusion is in fact an equality, and π is special unipotent attached to $\mathcal{O}^\vee$.

For classical Lie algebras (Section 1.2), the orbit $d(\mathcal{O})$ is roughly parametrized by the *transpose* of the partition parametrizing $\mathcal{O}$. (Given a partition of n, we may consider it as a Young diagram — i.e. a left justified array of boxes whose length decreases down rows — and then the transpose is the flip about the obvious diagonal.) This is exactly right in Type A, but can't be quite right in other types since the transpose of an element in, say, $\Pi_B(2n+1)$ does not belong to $\Pi_C(2n)$. Some minor refinement is necessary which we now describe.

First we recall the "X-collapse" operation on partitions ([CM, Chapter 6], for example). Recall the partial order on $\Pi(n)$ arising from the closure order on nilpotent orbits for $\mathfrak{sl}(n,\mathbb{C})$. Given a partition $\lambda \in \Pi(2n)$, its C- (resp. D-)collapse λ_C (resp. λ_D) is defined to be the largest partition in $\Pi_C(2n)$ (resp. $\Pi_D(2n)$) which is less than or equal to λ in the partial order on $\Pi(2n)$. The B-collapse λ_B of a partition $\lambda \in \Pi(2n+1)$ is defined in the same way.

Now we can give the computation of d in terms of partitions. If $\lambda \in \Pi_D(2n)$, then $d(\lambda)$ is obtained by transposing λ and then taking the D-collapse. (We have no occasion to keep track of orbits of $\mathrm{SO}(n,\mathbb{C})$.) If $\lambda \in \Pi_C(2n)$, the $d(\lambda)$ is obtained by first adding 1 to the largest part of λ, then taking the B-collapse, then transposing, and finally taking the B-collapse again. If $\lambda \in \Pi_B(2n+1)$ the $d(\lambda)$ is obtained by first removing one from the smallest part of λ, then taking the C-collapse, then transposing, and finally taking the C-collapse again.

1.7. **A dual operation: adding a row to a nilpotent orbit.** In view of Corollary 1.5 and Theorem 1.6, it's natural to ask it we can find an operation dual to adding a column to a nilpotent orbits. More precisely, consider an irreducible dual pair $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ with complexified Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$. We would like to find a map $\theta^\vee$ that makes the following diagram commute,

$$\begin{array}{ccc} \mathbf{N}(\mathfrak{g}) & \xrightarrow{\theta} & \mathbf{N}(\mathfrak{g}') \\ \downarrow d & & \downarrow d \\ \mathbf{N}(\mathfrak{g}^\vee) & \xrightarrow[\theta^\vee]{} & \mathbf{N}((\mathfrak{g}')^\vee). \end{array} \tag{2}$$

The map θ can be combinatorially defined (Section 1.5 and Remark 1.3), and we explained the combinatorial computation of d in Section 1.6. So investigating the existence of $\theta^\vee$ is tractable. Of course its essentially trivial in Type A, and so we start there.

Suppose $n \leq m$ and define

$$\theta^\vee \;:\; \Pi(n) \longrightarrow \Pi(m)$$

by augmenting a given partition of n by the part $m-n$; i.e. $\theta^\vee$ adds a row of length $m-n$ to n. Using that parametrizations of Section 1.2, this gives us a map

$$\theta^\vee \;:\; \mathbf{N}(\mathfrak{sl}(n,\mathbb{C})) \longrightarrow \mathbf{N}(\mathfrak{sl}(m,\mathbb{C})).$$

The following proposition is obvious.

Proposition 1.8. *Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{g}' = \mathfrak{sl}(m,\mathbb{C})$ and suppose $n \leq m$. Then for all $\mathcal{O} \in \mathbf{N}(\mathfrak{g})$,*

$$\theta^\vee(d(\mathcal{O})) = d(\theta(\mathcal{O})).$$

Corollary 1.9. *Retain the notation and hypotheses of Proposition 1.8. Consider an irreducible dual pair $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ whose complexified Lie algebras are equal to $(\mathfrak{g}, \mathfrak{g}')$. Suppose $\pi \in \mathrm{Irr}(\widetilde{G}_{\mathbb{R}})$ is a special unipotent representation attached to $\mathcal{O}^\vee$ (Definition 0.1). Then*

$$\mathrm{AV}(\mathrm{Ann}(\theta(\pi))) \subset \overline{d[\theta^\vee(\mathcal{O}^\vee))]}.$$

Proof. Since π is special unipotent attached to $\mathcal{O}^\vee$, Theorem 1.6 together with Corollary 1.5 gives that

$$\mathrm{AV}(\mathrm{Ann}(\theta(\pi))) \subset \overline{\theta(d(\mathcal{O}^\vee))}.$$

Now since d^2 is the identity, Proposition 1.8 implies

$$\theta(d(\mathcal{O}^\vee)) = [d\circ d\circ \theta\circ d](\mathcal{O}^\vee) = [d\circ \theta^\vee d\circ d](\mathcal{O}^\vee) = d(\theta^\vee(\mathcal{O}^\vee)),$$

and the corollary then follows. ☐

Since the collapse procedure is combinatorially a little complicated, defining $\theta^\vee$ outside of Type A in Equation (2) is also complicated outside of Type A. But there is one easy case that it entirely adequate for our applications.

Proposition 1.10. *Fix $n \leq m$ and consider of pair of complex Lie algebras $(\mathfrak{g}, \mathfrak{g}')$ of the form*

$$(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{sp}(2n,\mathbb{C}), \mathfrak{so}(2m,\mathbb{C})) \text{ or } (\mathfrak{so}(2n,\mathbb{C}), \mathfrak{sp}(2m,\mathbb{C}))$$

Fix an element $\mathcal{O} \in \mathbf{N}(\mathfrak{g})$ (Notation 1.1), and suppose that the corresponding partition λ has all parts occurring an even number of times (cf. Example 1.4). Consider $\mathcal{O}^\vee = d(\mathcal{O}) \in \mathbf{N}(\mathfrak{g}^\vee)$ and let $\lambda^\vee$ denote its corresponding partition. Then the partition obtained by adding a *single part ("row") of length $2m-2n$ to $\lambda^\vee$ is of the appropriate type to define an orbit $\theta^\vee(\mathcal{O}^\vee) \in \mathbf{N}((\mathfrak{g}')^\vee)$. Moreover*

$$\theta^\vee(d(\mathcal{O})) = d(\theta(\mathcal{O})).$$

The proof is a simple exercise in the combinatorial definitions. The assumption that all parts of the partition occur with even multiplicity simplifies matters enormously.

Corollary 1.11. *Retain the notation and hypothesis of Proposition 1.10. Consider an irreducible dual pair $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ whose complexified Lie algebras are equal to $(\mathfrak{g}, \mathfrak{g}')$. Suppose $\pi \in \mathrm{Irr}(\widetilde{G}_{\mathbb{R}})$ is a special unipotent representation attached to $\mathcal{O}^\vee$ (Definition 0.1; the assumption of the proposition guarantee that $\mathcal{O}^\vee$ is even). Then*

$$\mathrm{AV}(\mathrm{Ann}(\theta(\pi))) \subset \overline{d[\theta^\vee(\mathcal{O}^\vee)]}.$$

Proof. This follows just as Corollary 1.9 did. (Here one must also use the fact that $\mathcal{O}$ is special and that d^2 is the identity on special orbits.) □

1.8. **The infinitesimal character correspondence.** Fix a reductive dual pair $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ as in the introduction. Fix Cartans $\mathfrak{h}$ and $\mathfrak{h}'$ in $\mathfrak{g}$ and $\mathfrak{g}'$ and let W and W' denote the corresponding Weyl groups. Write

$$\mathrm{ic} \;:\; \mathrm{Irr}(\widetilde{G}_{\mathbb{R}}) \longrightarrow \mathfrak{h}^*/W,$$

for the infinitesimal character map. Fix $\pi \in \mathrm{Irr}(\widetilde{G}_{\mathbb{R}})$. Przebinda [Pr2] proved that then the infinitesimal character of $\theta(\pi)$ depends only on the infinitesimal character of π. More precisely, there is a map

$$\theta_{\mathrm{ic}} \;:\; \mathfrak{h}^*/W \longrightarrow (\mathfrak{h}')^*/W'$$

such that

$$\mathrm{ic}(\theta(\pi)) = \theta_{\mathrm{ic}}(\mathrm{ic}(\pi)),$$

whenever $\theta(\pi) \neq 0$. Moreover θ_{ic} depends only of the pair $(\mathfrak{g}, \mathfrak{g}')$ of complexified Lie algebras.

The main point is that the map $\theta^\vee$ of Section 1.5 computes θ_{ic} for the kinds of infinitesimal characters that arise as those of special unipotent representations of interest to us here.

Proposition 1.12. *Recall the notation $\chi(\mathcal{O}^\vee)$ appearing in Definition 0.1.*

(a) *Fix complex Lie algebras $(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{sl}(n,\mathbb{C}), \mathfrak{sl}(m,\mathbb{C}))$. Let $\mathcal{O}^\vee$ denote an even nilpotent orbit for $\mathfrak{g}^\vee \simeq \mathfrak{sl}(n,\mathbb{C})$. The condition that $\mathcal{O}^\vee$ is even means all of its rows have the same parity, say ϵ. Suppose $m-n$ is positive and matches the parity of ϵ. Then*

$$\theta_{\mathrm{ic}}(\chi(\mathcal{O}^\vee)) = \chi(\theta^\vee(\mathcal{O}^\vee)).$$

(b) *Retain the notation and assumptions of Proposition 1.10.*

$$\theta_{\mathrm{ic}}(\chi(\mathcal{O}^\vee)) = \chi(\theta^\vee(\mathcal{O}^\vee)).$$

Proof. This is a simple exercise using [Pr2] and the well-known computation of $\chi(\mathcal{O}^\vee)$ ([CM, Section 6.3] for example). □

1.9. Proof of Theorem 0.3.

Theorem 1.13. (a) *Retain the setting of Corollary 1.9. Then* $\theta(\pi)$ *is either zero or special unipotent attached to* $\theta^\vee(\mathcal{O}^\vee)$.

(b) *Retain the setting of Corollary 1.11. Then* $\theta(\pi)$ *is either zero or special unipotent attached to* $\theta^\vee(\mathcal{O}^\vee)$.

Proof. This follows immediately from Proposition 1.12 and Corollaries 1.7, 1.9, and 1.11. □

If π is a special unipotent representation of $\mathrm{O}^*(2n)$ or $\mathrm{Sp}(p,q)$, then the discussion in Section 3 shows that $\mathrm{AV}(\mathrm{Ann}(\pi))$ is the closure of an orbit parametrized by a partition all of whose parts occur an equal number of times. So the hypotheses of Theorem 0.3 imply those for Theorem 1.13(b), and the theorem thus follows once we note that Proposition 1.10 implies that

$$d(\theta(d(\mathcal{O}^\vee))) = \theta^\vee(\mathcal{O}^\vee).$$

□

2. UNIPOTENT REPRESENTATIONS OF $\mathrm{U}(p,q)$.

The purpose of this section is to give a very precise conjectural description of the theta lifts of unipotent representations of $\mathrm{U}(p,q)$ sharpening the conclusion of Theorem 1.13(a).

Theorem 2.1. *Let* $G_\mathbb{R} = \mathrm{U}(p,q)$ *and* $n = p+q$. *Fix an even nilpotent orbit* $\mathcal{O}^\vee$ *for* $\mathfrak{g}^\vee$ *and set* $\mathcal{O} = d(\mathcal{O}^\vee)$ *(Notation as in Section 1.6). Suppose* $\mathcal{O}\cap\mathfrak{p}$ *is nonempty. Let* $\mathrm{Unip}_{p,q}(\mathcal{O}^\vee)$ *denote the set of special unipotent representations of* $\mathrm{U}(p,q)$ *attached to* $\mathcal{O}^\vee$ *(Definition 0.1). Then there is a bijection*

$$\mathrm{Unip}_{p,q}(\mathcal{O}^\vee) \longrightarrow \mathrm{Irr}(\mathcal{O}\cap\mathfrak{p})$$

mapping $\pi \in \mathrm{Unip}_{p,q}(\mathcal{O}^\vee)$ *to the dense orbit in* $\mathrm{AV}(\pi)$ *(Section 1.4). In particular, there is a unique such orbit; i.e.* $\mathrm{AV}(\pi)$ *is irreducible. As a matter of notation, we will write* $\pi(\mathcal{O}_K)$ *for the special unipotent representation corresponding to* $\mathcal{O}_K \in \mathrm{Irr}(\mathcal{O}\cap\mathfrak{p})$.

Proof. The paper [BV2] essentially establishes the theorem. The modifier "essentially" is required since that reference makes no mention of associated varieties; see [T1, Section 4] for this. □

We now define $\pi(\mathcal{O}_K)$ in terms of cohomological induction. In order to do so, we must first recall the well-known parametrization of $\mathrm{Irr}(\mathcal{O}\cap\mathfrak{p})$. A signed tableau of signature (p,q) is an (equivalence class of) signed Young diagram of size $p+q$ whose boxes are filled with p plus signs and q minus signs so that signs alternate across rows; two such diagrams are said to be equivalent if they differ by interchanging rows of equal length. We write $\Pi(p,q)$ for the set of all such diagram and $\Pi(p,q;\lambda)$ for those whose shape coincides with that of $\lambda \in \Pi(p+q)$ when viewed a Young diagram. If $\mathcal{O}$ is parametrized by λ, then $\mathrm{Irr}(\mathcal{O}\cap\mathfrak{p})$ is parametrized by $\Pi(p,q;\lambda)$; see [CM, Chapter 9].

Recall that K conjugacy classes of τ-stable parabolics in $\mathfrak{g}$ are parametrized by ordered tuples $(p_1,q_1),\dots,(p_r,q_r)$ so that $\sum_i p_i = p$ and $\sum_i q_i = q$. The parabolic $\mathfrak{q} = \mathfrak{l}\oplus\mathfrak{u}$ corresponding to such a tuple has Levi factor $\mathfrak{l}$ which satisfies

$$\mathfrak{l}_\mathbb{R} = \mathfrak{q}\cap\bar{\mathfrak{q}} \simeq \mathfrak{u}(p_1,q_1)\oplus\cdots\oplus\mathfrak{u}(p_r,q_r).$$

More details may be found in [T1, Section 3].

Given $\mathcal{O}_K \in \mathrm{Irr}(\mathcal{O} \cap \mathfrak{p})$ parametrized by $\lambda_\pm \in \Pi(p,q;\lambda)$, let p_i denote the number of plus signs in its ith column; likewise let q_i denote the number of minus signs in the ith column. Denote the corresponding τ-stable parabolic by $\mathfrak{q}(\mathcal{O}_K) = \mathfrak{l}(\mathcal{O}_K) \oplus \mathfrak{u}(\mathcal{O}_K)$ (or just $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ when the context is clear). Let $L_\mathbb{R}$ denote the analytic subgroup of $G_\mathbb{R}$ with Lie algebra $\mathfrak{l}_\mathbb{R}$. Consider the one-dimensional representation $\mathbb{C}_\eta$ of $L_\mathbb{R}$ on the dual space $\wedge^{\mathrm{top}}(\mathfrak{u}^*)$. The condition on the parity of the columns guarantees that the square-root character $C_\lambda := \mathbb{C}_{\sqrt{\eta}}$ factors to $L_\mathbb{R}$. (See the discussion after Definition 1.31 in [V3] for instance.) Define

$$\pi(\mathcal{O}_K) = A_\mathfrak{q}(\lambda), \tag{3}$$

where $A_\mathfrak{q}(\lambda)$ denotes the derived functor module defined (for instance) in [KnV, Chapter 5]. The normalization is arranged so that $\pi(\mathcal{O}_K)$ has infinitesimal character that matches the trivial representation of $L_\mathbb{R}$. In the terminology of [KnV, Definition 0.52] this module is exactly on the "edge" of the weakly fair range; i.e. for all roots $\alpha \in \Delta(\mathfrak{u})$, the inner product of α with $\lambda + \rho(\mathfrak{u})$ is zero. [BV2] proves that each $\pi(\mathcal{O}_K)$ is nonzero and [V2] proves that they are all unitary, thus verifying the prediction of Conjecture 0.2 in this case.

Here is the definition we need to investigate the theta lifts of the representations $\pi(\mathcal{O}_K)$. (It is probably best to read the definition in conjunction with the example that follows it.)

Definition 2.2. Consider $(G_\mathbb{R}, G'_\mathbb{R}) = (\mathrm{U}(p,q), \mathrm{U}(r,s))$. Set $n = p+q$, $m = r+s$, and assume $n \leq m$. Write $(\mathfrak{g}, \mathfrak{g}')$ for the corresponding complexified Lie algebras. Fix $\mathcal{O} \in \mathbf{N}(\mathfrak{g})$ parametrized by $\lambda \in \Pi(n)$. Let $\mathcal{O}' = \theta(\mathcal{O})$ (cf. Proposition 1.2), the element of $\mathbf{N}(\mathfrak{g}')$ parametrized by the partition λ' obtained from λ by adding a column of length $m-n$. Fix $\mathcal{O}_K \in \mathrm{Irr}(\mathcal{O} \cap \mathfrak{p})$ and suppose $\mathcal{O}_K$ is parametrized by the signed tableau $\lambda_\pm$.

We say that *the pair* $(\mathrm{U}(p,q), \mathrm{U}(r,s))$ *is in the stable range with respect to* $\mathcal{O}_K$ if there exists an orbit $\mathcal{O}'_K \in \mathrm{Irr}(\mathcal{O}' \cap \mathfrak{p}')$ parametrized by $\lambda'_\pm$ satisfying the following condition: there is some representative of $\lambda'_\pm$ whose subtableau of shape λ is a representative for $\lambda_\pm$. In this case, it is easy to see that $\lambda'_\pm$ (and hence $\mathcal{O}'_K$) is unique. We write $\mathcal{O}'_K = \theta(\mathcal{O}_K)$. (See Remark 2.4.)

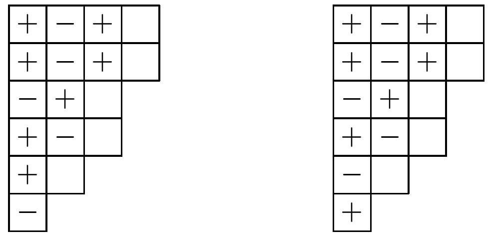
The terminology is consonant with the usual notion of the stable range for a dual pair (which in this case amounts to $\min(r,s) \geq p+q$): if $(\mathrm{U}(p,q), \mathrm{U}(r,s))$ is in the usual stable range then it is in in the stable range for all orbits $\mathcal{O}_K$; and conversely, the pair is in the usual stable range only if it is in the stable range for the zero orbit $\mathcal{O}_K$ of $\mathrm{U}(p,q)$.

Example 2.3. Consider $(p,q) = (7,5)$ and let $\mathcal{O}_K$ be the orbit parametrized by

$$\lambda_\pm \quad = \quad \begin{array}{|c|c|c|}\hline + & - & + \\ \hline + & - & + \\ \hline - & + \\ \cline{1-2} + & - \\ \cline{1-2} + \\ \cline{1-1} - \\ \cline{1-1}\end{array} \qquad ,$$

and so the complex orbit $\mathcal{O}$ is parametrized by the partition $\lambda = (3,3,2,2,1,1)$. Suppose that $r+s = 17$ and thus the complex orbit $\mathcal{O}' = \theta(\mathcal{O})$ obtained by adding a column of length 5 to λ is parametrized by $\lambda' = (4,4,3,3,2,1)$. Thus the signed partition $\lambda'_\pm$ parametrizing

$\mathcal{O}'_K = \theta(\mathcal{O}_K)$ is obtained by filling the vacancies in one of the following diagrams



This is possible only if $(r,s) = (8,9)$ or $(9,8)$ in which case such a filling is unique. (It is easy to see the unicity of such fillings in the general case.) In the terminology of Definition 2.2, the pair $(\mathrm{U}(7,5), \mathrm{U}(r,s))$ with $r+s = 17$ is in the stable range for $\mathcal{O}_K$ if and only if $(r,s) = (8,9)$ or $(9,8)$.

Remark 2.4. The notation $\theta(\mathcal{O}_K)$ is potentially dangerous inasmuch as it resembles the notation $\theta(\mathcal{O})$. According to Proposition 1.2, the latter notation had both a combinatorial and geometric interpretation; but Definition 2.2 gives only a combinatorial definition of $\theta(\mathcal{O}_K)$. Fortunately there is a corresponding geometric interpretation which was observed by a number of people. More precisely, let $\mathcal{O}_K^{\min}$ denote the associated variety of the oscillator representation of $\mathrm{Mp}(2N,\mathbb{R})$ used to define the theta correspondence. Write $\mathfrak{s}$ for the -1 eigenvalue of a complexified Cartan involution for $\mathrm{Mp}(2N,\mathbb{R})$ on $\mathfrak{sp}(2n,\mathbb{C})$. Recall the notational conventions of Section 1.1; so $\mathfrak{p}, \mathfrak{p}' \subset \mathfrak{s}$. Write μ_K for the restriction to $\overline{\mathcal{O}_K^{\min}}$ of the natural projection of $\mathfrak{s}^*$ to $\mathfrak{p}^*$ and likewise for μ'_K. Then for a particular choice of the oscillator defining the θ correspondence, $\theta(\mathcal{O}_K)$ is dense in $\mu'_K(\mu_K^{-1}(\overline{\mathcal{O}_K}))$. (I learned this from unpublished work of Shu-Yen Pan. The paper [DKP2] is also relevant.) If we make the other choice of oscillator defining θ, the combinatorial definition of $\theta(\mathcal{O}_K)$ must be modified to coincide with the geometric one: instead of adding a column of singed entries on the right (as in Example 1.4), the column is instead added on the left. In Conjecture 2.5 (and Conjecture 3.5 below) *a specific choice of oscillator is implicitly fixed* so that the combinatorial definition of $\theta(\mathcal{O}_K)$ in Definitions 2.2 and 3.2 is compatible with the natural geometric one.

Here is how unipotent representations of $\mathrm{U}(p,q)$ should behave under theta lifting.

Conjecture 2.5. *Let $(G_{\mathbb{R}}, G'_{\mathbb{R}}) = (\mathrm{U}(p,q), \mathrm{U}(r,s))$. Fix an even complex orbit $\mathcal{O}^\vee \in \mathbf{N}(\mathfrak{g})$ and set $\mathcal{O} = d(\mathcal{O}^\vee)$ (Section 1.6), and suppose that the orbit $\theta^\vee(\mathcal{O}^\vee) \in \mathbf{N}((\mathfrak{g}')^\vee)$ (Section 1.7) is also even. Fix an orbit $\mathcal{O}_K \in \mathrm{Irr}(\mathcal{O} \cap \mathfrak{p})$ and suppose the pair is in the stable range for $\mathcal{O}_K$ (Definition 2.2). Recall the choice of oscillator discussed in Remark 2.4. Then*

$$\theta(\pi(\mathcal{O}_K)) = \pi(\theta(\mathcal{O}_K)).$$

Remark 2.6. The validity of the conjecture would give a mechanism to obtain all special unipotent representations of $\mathrm{U}(p,q)$ as iterated lifts of the trivial representation. More precisely, given $\mathcal{O}_K$ we may build a sequence of orbits $\mathcal{O}_{\text{zero}} = \mathcal{O}_K^{(1)}, \mathcal{O}_K^{(2)}, \ldots, \mathcal{O}_K^{(k)} = \mathcal{O}_K$ by incrementally peeling away the longest possible chain of row-ends from the right side of the tableau parametrizing $\mathcal{O}_K$. This is most easily described by means of an example. In the following chain of tableau, the left-most one parametrizes $\mathcal{O}_K$ and each successive tableau is obtained by peeling off row-ends (indicated with primes) from the previous tableau:

$$
\begin{array}{|c|c|c|c|c|}
\hline + & - & + & - & +' \\
\hline + & - & + & - & +' \\
\hline - & + & - & +' \\
\cline{1-4} + & - & + & -' \\
\cline{1-4} + & -' \\
\cline{1-2} + & -' \\
\cline{1-2} +' \\
\cline{1-1} -' \\
\cline{1-1}
\end{array}
\qquad
\begin{array}{|c|c|c|c|}
\hline + & - & + & -' \\
\hline + & - & + & -' \\
\hline - & + & -' \\
\cline{1-3} + & - & +' \\
\cline{1-3} +' \\
\cline{1-1} +' \\
\cline{1-1}
\end{array}
\qquad
\begin{array}{|c|c|c|}
\hline + & - & +' \\
\hline + & - & +' \\
\hline - & +' \\
\cline{1-2} + & -' \\
\cline{1-2}
\end{array}
\qquad
\begin{array}{|c|c|}
\hline + & -' \\
\hline + & -' \\
\hline -' \\
\cline{1-1} +' \\
\cline{1-1}
\end{array}
\qquad
\begin{array}{|c|}
\hline + \\
\hline + \\
\hline
\end{array}
$$

(A different choice of oscillator would necessitate peeling from the left; see Remark 2.4.) Let $\mathcal{O}_K^{(1)}$ denote the orbit parametrized by the last tableau (so it is necessarily the zero orbit), let $\mathcal{O}_K^{(2)}$ denote the orbit parametrized by the second to last tableau, and so on; we end with $\mathcal{O}_K^{(k)} = \mathcal{O}_K$. Let (p_i, q_i) denote the signature of the ith tableau, so that $\mathcal{O}_K^{(i)}$ is an orbit for $\mathrm{U}(p_i, q_i)$. Consider the sequential θ liftings

$$\mathrm{Irr}(\mathrm{U}(p_1, q_1)) \xrightarrow{\theta_1} \mathrm{Irr}(\mathrm{U}(p_2, q_2)) \xrightarrow{\theta_2} \cdots \xrightarrow{\theta_{k-1}} \mathrm{Irr}(\mathrm{U}(p_k, q_k)).$$

It is easy to check that $\mathcal{O}_K^{(i)}$ is in the stable range for the pair $(\mathrm{U}(p_i, q_i), \mathrm{U}(p_{i+1}, q_{i+1}))$. Since $\mathcal{O}_K^{(1)}$ is the zero orbit, the validity of the conjecture would imply that

$$\pi(\mathcal{O}_K) = [\theta_{k-1} \circ \cdots \circ \theta_2 \circ \theta_1](\mathbb{1}_{p_1,q_1}),$$

where $\mathbb{1}_{p_1,q_1}$ denotes the trivial representation of $\mathrm{U}(p_1, q_1)$.

Notice that Theorem 1.13(a) implies that in the setting of Conjecture 2.5,

$$\theta(\mathrm{Unip}_{p,q}(\mathcal{O}^\vee)) \subset \mathrm{Unip}_{r,s}(\theta^\vee(\mathcal{O}^\vee)).$$

The next theorem shows how to make the inclusion an equality in the stable range.

Theorem 2.7. *Fix* $\mathrm{U}(r, s)$ *and fix* $n \leq \min(r, s)$. *Set* $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ *and* $\mathfrak{g}' = \mathfrak{gl}(r+s, \mathbb{C})$. *Fix an even complex orbit* $\mathcal{O}^\vee \in \mathbf{N}(\mathfrak{g})$ *and set* $\mathcal{O} = d(\mathcal{O}^\vee) \in \mathbf{N}(\mathfrak{g}')$ *(Section 1.6), and suppose that the orbit* $\theta^\vee(\mathcal{O}^\vee) \in \mathbf{N}((\mathfrak{g}')^\vee)$ *(Section 1.7) is also even. Then*

$$\theta\left(\bigcup_{p+q=n} \mathrm{Unip}_{p,q}(\mathcal{O}^\vee)\right) = \mathrm{Unip}_{r,s}(\theta^\vee(\mathcal{O}^\vee)). \tag{4}$$

Proof. As remarked before above, Theorem 1.13(a) implies that the left-hand side of Equation (4) is a subset of the right-hand side. So it remains only to show that both sides are sets of the same cardinality. Since each pair $(\mathrm{U}(p, q), \mathrm{U}(r, s))$ is in the stable range, $\theta(\pi) \neq 0$ for all $\pi \in \mathrm{Irr}(\mathrm{U}(p, q))$. Thus the left-hand side of Equation (4) has cardinality equal to that of

$$\bigcup_{p+q=n} \mathrm{Unip}_{p,q}(\mathcal{O}^\vee).$$

By Theorem 2.1 and the combinatorial classification discussed above, the above set is in bijection with

$$\bigcup_{p+q=n} \Pi_A(p, q; \lambda), \tag{5}$$

where λ denotes the shape of $\mathcal{O}$. Meanwhile the right-hand side is in bijection with

$$\Pi_A(r,s;\lambda'), \tag{6}$$

where λ' is the shape of $\theta(\mathcal{O})$. But since $\min(r,s) \geq p+q$, the sets in (5) and (6) are clearly in bijection: in the notation of Definition 2.2, the map simply takes a tableau in (5) parametrizing an orbit $\mathcal{O}_K$ to the tableau in (6) parametrizing the orbit $\theta(\mathcal{O}_K)$ obtained by adding a a column of the appropriate signs. □

Notice that the validity of Conjecture 2.5 would imply Theorem 2.7. So the latter may be interpreted as evidence for the former.

In the setting of Theorem 2.7, the preservation of unitarity results of [Li1] allow one to deduce the unitarity of each element of the set $\mathrm{Unip}_{r,s}(\theta^\vee(\mathcal{O}^\vee))$ from the unitarity of the set $\theta_{p,q}(\mathrm{Unip}(\mathcal{O}^\vee))$. As remarked above, we know that every special unipotent representation of $\mathrm{U}(p,q)$ is unitary, so we obtain nothing new. By contrast, the parallel theory of the next section provides new unitarity results.

3. UNIPOTENT REPRESENTATIONS OF $\mathrm{Sp}(p,q)$ AND $\mathrm{O}^*(2n)$

The theory of unipotent representations of $\mathrm{Sp}(p,q)$ and $\mathrm{O}^*(2n)$ closely mirrors that of $\mathrm{U}(p,q)$.

Theorem 3.1. *Let $G_\mathbb{R} = \mathrm{Sp}(p,q)$ and $n = p+q$. Fix an even nilpotent orbit $\mathcal{O}^\vee$ for $\mathfrak{g}^\vee$ and set $\mathcal{O} = d(\mathcal{O}^\vee)$ (Notation as in Section 1.6). Let $\mathrm{Unip}_{p,q}(\mathcal{O}^\vee)$ denote the set of special unipotent representations attached to $\mathcal{O}^\vee$ (Definition 0.1). Suppose $\mathcal{O}\cap\mathfrak{p}$ is nonempty. Then there is a bijection*

$$\mathrm{Unip}_{p,q}(\mathcal{O}^\vee) \longrightarrow \mathrm{Irr}(\mathcal{O}\cap\mathfrak{p})$$

mapping $\pi \in \mathrm{Unip}(\mathcal{O}^\vee)$ to the dense orbit in $\mathrm{AV}(\pi)$ (Section 1.4). In particular, there is a unique such orbit; i.e. $\mathrm{AV}(\pi)$ is irreducible. As a matter of notation, we will write $\pi(\mathcal{O}_K)$ for the special unipotent representation corresponding to $\mathcal{O}_K \in \mathrm{Irr}(\mathcal{O}\cap\mathfrak{p})$.

The same conclusion holds for $G_\mathbb{R} = \mathrm{O}^*(2n)$. As a matter of notation, We write $\mathrm{Unip}_{2n}(\mathcal{O}^\vee)$ for the set of special unipotent representations of $\mathrm{O}^*(2n)$ attached to $\mathcal{O}^\vee$.

Proof. Theorems 6 and 10 of [Mc1] prove the existence of the bijection. Here is a sketch of the associated variety. The main results of [V4] show that each $\mathrm{AV}(\pi)$ is irreducible. The surjectivity of the assignment $\pi \mapsto \mathrm{AV}(\pi)$ follows by a counting argument of the form given in Theorem 3.3.1 of [T3]. □

Unlike the case of $\mathrm{U}(p,q)$, the unipotent representation $\pi(\mathcal{O}_K)$ of $\mathrm{Sp}(p,q)$ and $\mathrm{O}^*(2n)$ need not be weakly fair derived functor modules. (The paper [T2] determines exactly which of them are.) It seems likely that all are obtained by a kind of analytic continuation of such derived functor modules however.

As in the previous section, we need to recall the combinatorics of the sets $\mathrm{Irr}(\mathcal{O}\cap\mathfrak{p})$ appearing in Theorem 3.1. So let $G_\mathbb{R} = \mathrm{Sp}(p,q)$ and retain the notation of the theorem. The condition that $\mathcal{O}\cap\mathfrak{p}$ be nonempty implies that all parts of the partition λ parameterizing $\mathcal{O}$ appear with even multiplicity. Let $\Pi_C(2p,2q;\lambda)$ denote the subset of $\Pi(2p,2q,\lambda)$ (with notation as in Section 2) consisting of those signed tableau for which each chunk of even rows of a fixed length has an equal number of rows beginning with plus and with minus. Then $\Pi_C(2p,2q;\lambda)$ parametrizes $\mathrm{Irr}(\mathcal{O}\cap\mathfrak{p})$; see [CM, Theorem 9.3.5].

Next let $G_{\mathbb{R}} = \mathrm{O}^*(2n)$ and fix other notation as in Theorem 3.1. Again let λ denote the partition parametrizing $\mathcal{O}$. As before, the condition that $\mathcal{O} \cap \mathfrak{p}$ be nonempty implies that all parts of λ appear with even multiplicity. Let $\Pi_D(*, *; \lambda)$ denote the subset of

$$\Pi(*, *; \lambda) = \bigcup_{p+q=2n} \Pi(p, q; \lambda)$$

consisting of those partitions for for which each chunk of odd rows of a fixed length has an equal number of rows beginning with plus and with minus. Then $\Pi_D(*, *; \lambda)$ parametrizes $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{p})$; see [CM, Theorem 9.3.4].

Definition 3.2. Consider $(G_{\mathbb{R}}, G'_{\mathbb{R}}) = (\mathrm{Sp}(p, q), \mathrm{O}^*(2m))$. Set $n = p+q$ and assume $n \leq m$. Write $(\mathfrak{g}, \mathfrak{g}')$ for the corresponding complexified Lie algebras. Fix $\mathcal{O} \in \mathbf{N}(\mathfrak{g})$ parametrized by $\lambda \in \Pi_C(2n)$ in which all parts occur with even multiplicity. Let $\mathcal{O}' = \theta(\mathcal{O})$ (cf. Example 1.4), the element of $\mathbf{N}(\mathfrak{g}')$ parametrized by the partition λ' obtained from λ by adding a column of length $2m - 2n$. Fix $\mathcal{O}_K \in \mathrm{Irr}(\mathcal{O} \cap \mathfrak{p})$ and suppose $\mathcal{O}_K$ is parametrized by the signed tableau $\lambda_{\pm}$.

We say that *the pair* $(\mathrm{Sp}(p, q), \mathrm{O}^*(2m))$ *is in the stable range with respect to* $\mathcal{O}_K$ if there exists an orbit $\mathcal{O}'_K \in \mathrm{Irr}(\mathcal{O}' \cap \mathfrak{p}')$ parametrized by $\lambda'_{\pm}$ satisfying the following condition: there is some representative of $\lambda'_{\pm}$ whose subtableau of shape λ is a representative for $\lambda_{\pm}$. In this case, it is easy to see that $\lambda'_{\pm}$ (and hence $\mathcal{O}'_K$) is unique. We write $\mathcal{O}'_K = \theta(\mathcal{O}_K)$.

We say that *the pair* $(\mathrm{Sp}(p, q), \mathrm{O}^*(2m))$ *is in the stable range with respect to* $\mathcal{O}$ if it is in the stable range with respect to every $\mathcal{O}_K \in \mathrm{Irr}(\mathcal{O} \cap \mathfrak{p})$.

The obvious versions of these definitions apply to $(G_{\mathbb{R}}, G'_{\mathbb{R}}) = (\mathrm{O}^*(2n), \mathrm{Sp}(p, q))$ with $n \leq p + q$.

The terminology is again compatible with the usual notion of the stable range: a pair of the above type is in the usual stable range if and only if it is in the stable range for all complex orbits $\mathcal{O}$.

Remark 3.4. The analogous version of Remark 2.4 applies here.

Here is the analog of Conjecture 2.5.

Conjecture 3.5. *Let* $(G_{\mathbb{R}}, G'_{\mathbb{R}}) = (\mathrm{Sp}(p, q), \mathrm{O}^*(2n))$ *or* $(\mathrm{O}^*(2m), \mathrm{Sp}(p, q))$. *Fix an even complex orbit* $\mathcal{O}^\vee \in \mathbf{N}(\mathfrak{g})$ *and set* $\mathcal{O} = d(\mathcal{O}^\vee)$ *(Section 1.6). Fix* $\mathcal{O}_K \in \mathcal{O} \cap \mathfrak{p}$ *and suppose the pair is in the stable range for* $\mathcal{O}_K$ *(Definition 3.2). Recall the choice of oscillator discussed in Remark 2.4. Then*

$$\theta(\pi(\mathcal{O}_K)) = \pi(\theta(\mathcal{O}_K)).$$

Remark 3.6. Just an in Remark 2.6, the validity of the conjecture would precisely dictate (in terms of an analogous "peeling" procedure) how each unipotent representation is an iterated lift from the trivial representation.

Next we have the analog of Theorem 2.7.

Theorem 3.7. *Recall the notational conventions of Section 1.1 and the notation of Theorem 3.1.*

(1) *Fix* $G'_{\mathbb{R}} = \mathrm{Sp}(p, q)$*, fix* $n \leq \min(p, q)$*, and let* $G_{\mathbb{R}} = \mathrm{O}^*(2n)$*. Fix an orbit* $\mathcal{O}^\vee \in \mathbf{N}(\mathfrak{g})$*, set* $\mathcal{O} = d(\mathcal{O}^\vee) \in \mathbf{N}(\mathfrak{g}')$ *(Section 1.6), and suppose that* $\mathcal{O} \cap \mathfrak{p}$ *is nonempty. Then*

$$\theta\left(\mathrm{Unip}_{2n}(\mathcal{O}^\vee)\right) = \mathrm{Unip}_{p,q}(\theta^\vee(\mathcal{O}^\vee)). \tag{7}$$

In particular, since nonzero unitary representations lift to nonzero unitary representations in the stable range [Li1]*, the validity of Conjecture 0.2 for* $\mathrm{O}^*(2n)$ *implies it for* $\mathrm{Sp}(p,q)$*.*

(2) *Fix* $G'_{\mathbb{R}} = \mathrm{O}^*(2n)$*, and fix* m *such that* $2m \leq n$*. Set* $\mathfrak{g} = \mathfrak{sp}(2m,\mathbb{C})$ *and* $\mathfrak{g}' = \mathfrak{so}(2n,\mathbb{C})$*. Fix an orbit* $\mathcal{O}^\vee \in \mathbf{N}(\mathfrak{g})$*, set* $\mathcal{O} = d(\mathcal{O}^\vee) \in \mathbf{N}(\mathfrak{g}')$ *(Section 1.6), and suppose that* $\theta(\mathcal{O}) \cap \mathfrak{p}'$ *is nonempty. Then*

$$\theta\left(\bigcup_{p+q=m} \mathrm{Unip}_{p,q}(\mathcal{O}^\vee)\right) = \mathrm{Unip}_{2n}(\theta^\vee(\mathcal{O}^\vee)). \tag{8}$$

In particular, the validity of Conjecture 0.2 for the various groups $\mathrm{Sp}(p,q)$ *implies it for* $\mathrm{O}^*(2n)$*.*

Proof. This follows from counting considerations analogous to those treated in the proof of Theorem 2.7. We omit the details. □

As remarked above, the results of [He] may make it possible to drop the stable range hypothesis of Theorem 3.7 and thus prove Conjecture 0.2 for $\mathrm{Sp}(p,q)$ and $\mathrm{O}^*(2n)$ without additional hypotheses. In any event, the present Theorem 3.7 supplies some evidence for Conjecture 3.5.

Finally, we may complete the proof of Theorem 0.4.

Proof of Theorem 0.4. Suppose π is a special unipotent representation of $\mathrm{Sp}(p,q)$ attached to $\mathcal{O}^\vee$, a nilpotent orbit for $\mathfrak{g}^\vee = \mathfrak{so}(2p+2q+1,\mathbb{C})$ of the form described in Theorem 0.4. Set $2m = 2p+2q$. Let p be the largest part of the partition λ parametrizing $\mathcal{O}^\vee$. Since p is assumed to be greater than the sum of all other parts less than or equal to it, p must have multiplicity one. Since even parts must occur with even multiplicity in the partition parametrizing $\mathcal{O}^\vee$, p must be odd. Set $2m_1 = 2m+1-p$. We may assume that $m_1 \neq 0$; otherwise the theorem is trivial. Let λ_1 be the partition obtained from λ by deleting the part p. Since all even parts in λ occur with even multiplicity and p is odd, the same is true of λ'. So λ_1 parametrizes a nilpotent orbit $\mathcal{O}_1^\vee$ for $\mathfrak{so}(2m_1,\mathbb{C})$.

Consider the dual pair $(\mathrm{O}^*(2m_1), \mathrm{Sp}(p,q))$. We claim the pair is in the stable range, i. e. that $m \leq \min(p,q)$. Since π is attached to $\mathcal{O}^\vee$, $d(\mathcal{O}^\vee) \cap \mathfrak{p}$ is nonempty (by Theorem 3.1). From the combinatorial description of $d(\mathcal{O}) \cap \mathfrak{p}$ given above, we conclude that there is a signature $(2p,2q)$ tableau $\eta_\pm$ whose shape coincides with η, the shape of $d(\mathcal{O}^\vee)$. From the definition of d, one checks that that η has a column of length p. Let η_1 denote the partition obtained from η after removing a column of length p. Likewise let $\eta_{1,\pm}$ denote the signed tableau obtained from $\eta_\pm$ by deleting all row-ends. (So, in the notation of Definition 3.2, $\eta_\pm = \theta(\eta_{1,\pm})$.) Let (r,s) denote the signature of $\eta_{1,\pm}$. Without loss of generality suppose $r \geq s$; so $r \geq m_1$. Thus the number of odd rows of $\eta_{1,\pm}$ ending in plus exceed the number of odd rows ending in minus by $r-s$. Thus the removed row-ends from $\eta_\pm$ must have had at least $r-s$ minus signs in it. So the number of minus signs in $\eta_\pm$ is at least $s+(r-s)$ (the former from $\eta_{1,\pm}$, the latter from the deleted row-ends), and hence the number of minus signs q exceeds r and hence m_1. The number of plus signs p in $\eta_\pm$ is at least the number in $\eta_{1,\pm}$ which was r and $r \geq m_1$. In other words, $m_1 \leq \min(p,q)$. So indeed the pair is in the stable range.

According to Theorem 3.7, there is a special unipotent representation π_1 of $\mathrm{O}^*(2m_1)$ attached to $\mathcal{O}_1^\vee$ such that $\theta(\pi_1) = \pi$. If π_1 is trivial, we are finished. If not, we may repeat this argument inductively (inverting the roles of $\mathrm{Sp}(p,q)$ and $\mathrm{O}^*(2n)$ as necessary and arguing combinatorially as before) to conclude that π is an iterated stable-range lift from the trivial representation. Hence π is unitary and Theorem 0.4 follows. □

Remark 3.8. The paper [LPTZ] computes a large part of the $\mathrm{O}^*(2n)$-$\mathrm{Sp}(p,q)$ correspondence in terms of Langlands parameters. But since the Langlands parameters of special unipotent representations are obscure, their computations cannot be immediately applied to Conjecture 3.5. On the other hand, the Langlands parameters of special unipotent representation of $\mathrm{U}(p,q)$ are known from [T1]. But Paul's computations in [P1]–[P2], while substantial, are seemingly not enough to handle Conjecture 2.5.

4. Considerations for general dual pairs

Many of the above ideas work for arbitrary dual pairs, but there are some complications. Of course if one is interested only in Conjecture 0.2, then the only simple linear groups (apart from those treated here) that arise in dual pairs for which the Conjecture 0.2 is open are $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{O}(p,q)$.

First there are what one might call obvious combinatorial complications arising in extending Propositions 1.10 and 1.12 to more general setting. Unfortunately many case-by-case considerations enter. For any particular case, these issues are relatively easy to resolve (but sometimes difficult to state).

Next, Theorem 0.3 can fail in the general case. The first example is the complex dual pair $(\mathrm{O}(2,\mathbb{C}),\mathrm{Sp}(2n,\mathbb{C}))$ for $n \geq 2$ and π the trivial representation of $\mathrm{O}(2,\mathbb{C})$. Then it follows from [AB] that $\theta(\pi)$ is also the trivial representation. While $\theta(\pi)$ is special unipotent of course, it is *not* attached to $d(\theta(\mathcal{O}))$. (The associated variety of its annihilator is the zero orbit which is properly contained in $\theta(\mathrm{AV}(\mathrm{Ann}(\pi)))$.) It seems possible that this kind of failure is confined to the complex pair $(\mathrm{O}(2m,\mathbb{C}),\mathrm{Sp}(2n,\mathbb{C}))$.

Finally exhaustion statements like those in Theorems 2.7 and 3.7 (and their analogous conjectures outside the stable range) are much more subtle for disconnected groups. To take but one example, fix s even and consider the $4s$ characters of the orthogonal groups $\mathrm{O}(p,q)$ with $p+q=s$. Fix $m \geq s$. Then the $4s$ characters lift to $4s$ special unipotent representations of $\mathrm{Sp}(2m,\mathbb{R})$. It is an easy count to see that these $4s$ representations exhaust all special unipotent representations attached $\mathcal{O}_1 = d(\theta(\mathcal{O}))$ where $\mathcal{O}$ is the zero orbit for $\mathrm{O}(s,\mathbb{C})$. Up to this point, everything looks fine. Now fix $\mathrm{O}(r,s)$ with $r+s \geq 4m$ and even. Then the $4s$ special unipotent representations of $\mathrm{Sp}(2m,\mathbb{R})$ just constructed lift to $4s$ special unipotent representations of $\mathrm{O}(r,s)$ attached to $d(\theta(d(\mathcal{O}_1))$. But this time, these $4s$ representations do *not* exhaust all such special unipotent representations — there are many more not accounted for by this construction. One may hope that after tensoring with nontrivial characters, they do indeed exhaust such unipotent representations. But this appears rather subtle to prove and perhaps even a little optimistic.

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