UPSC Problem Set 3

Problem 1

Consider an infinite series whose nth term is $\pm (1/n)$, the $\pm$ signs being determined according to a pattern that repeats periodically in blocks of eight. [There are 2^8 possible patterns of which two examples are:

$$+ + - - - - + +$$

$$+ - - - + - - -$$

The first example would generate the series $1 + (1/2) - (1/3) - (1/4) - (1/5) - (1/6) + (1/7) + (1/8) + (1/9) + (1/10) - (1/11) - (1/12) - \cdots$

a) Show that a sufficient condition for the series to be conditionally convergent is that there be four " $+$ " signs and four " $-$ " signs in the block of eight.
b) Is the sufficient condition also necessary?

[Here "convergent" means "convergent to a finite limit"]

Problem 2

Let $z = x + iy$ be a complex number with x and y rational and with $|z| = 1$. Show that the number $|z^{2n} - 1|$ is rational for every integer n.

Problem 3

Evaluate the following integral:

$$\int_0^1 x^{-x}\, dx$$