Problem 5

Undergraduate Problem Solving Contest
due Febuary , 2017

February 15, 2017

1 Matrix Polynomials

Let $\mathbb{Z}_4^{2\times 2}$ represent the 2×2 matricies with entries in the integers modulo 4 ($\mathbb{Z}_4$). A *left* matrix polynomial on $\mathbb{Z}_4^{2\times 2}$ is a function from $\mathbb{Z}_4^{2\times 2}$ to $\mathbb{Z}_4^{2\times 2}$ of the form:

$$f(X) = A_d X^d + A_{d-1} X^{d-1} + \cdots + A_1 X + A_0$$

Where A_i are in $\mathbb{Z}_4^{2\times 2}$.

Q: *Find 2 left matrix polynomials f, g such that:*

1. For all $X \in \mathbb{Z}_4^{2\times 2}$, $f(X) = g(X) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
2. f, g are relatively prime. Ie there is no left matrix polynomial $h(x)$ (with degree ≥ 1) such that $f(x) = h(x) \cdot q_f(x)$ and $g(x) = h(x) \cdot q_g(x)$ for any left matrix polynomials $q_f(x), q_g(x)$.
3. f, g have degree ≥ 1.

1.1 example

An example left matrix polynomial of degree 2:

$$f(X) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X^2 + \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}$$

$$f(\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$