

WINNING WAYS¹

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January 23, 2008

This week we talk about winning strategies for some simple (and not so simple!) games. There will be many themes, but perhaps the simplest is exploiting symmetries whenever possible. Here's a challenge to get started:

Double Chess. The game is played the same as regular chess, but during each turn a player is required to take two ordinary chess moves. Prove that there exists a strategy that guarantees at least a draw.

For each of the following games, two players alternate turns. The winner is the last player who makes a legal move. See if you can find a winning strategy for one of the players. Try to prove that your strategy works.

1. Consider a $1 \times n$ board. At each turn, a player places a 2×1 domino on the board so that it does not overlap with other dominos already on the board.
2. Same as (1), but the board is $2 \times n$.
3. A set of 16 pennies is placed on a table. Two players take turns removing pennies. At each turn, a player must remove between 1 and 4 pennies (inclusive).
4. Can you generalize the previous game to other values (besides 16 and 4)?
5. Start with a rectangular chocolate bar which is 6×8 squares in size. A legal move is breaking a piece of chocolate along a single straight line bounded by the squares. For example, you can turn the original bar into a 6×2 piece and a 6×6 piece, and this latter piece can be turned into a 1×6 piece and a 5×6 piece. What about the general case (the starting bar is $m \times n$)?
6. Each player takes turns placing a penny on the surface of a rectangular table. No penny can touch a penny that is already on the table. The table starts out completely bare.
7. We start with a pile of 7 kittens and 10 puppies. Two players take turns; a legal move is removing any number of puppies or any number of kittens or an equal number of both puppies and kittens.
8. You start with an $n \times m$ grid of graph paper. Players take turns coloring red one previously uncolored unit edge of the grid (including the boundary). A move is legal as long as no closed path has been created.
9. A game is played with two players and an initial stack of n pennies ($n \geq 3$). The players take turns choosing one of the stacks of pennies on the table and splitting it into two stacks. When a player makes a move that causes all the stacks to be of height 1 or 2 at the end of his or her turn, that player wins. Which starting values of n are wins for each player?

¹Most of games below are shamelessly stolen from a list compiled by Paul Zeitz for the San Francisco Math Circle.

10. Start with several piles of beans. A legal move consists of removing one or more beans from a pile.

(a) Verify that this game is very easy to play if you start with just one pile, for example, of 17 beans.

(b) Likewise, if the game starts with two piles, the game is quite easy to analyze. Do it!

(c) But what if we start with three or more piles? For example, how do we play the game if it starts with three piles of 17, 11, and 8 beans, respectively? What about four piles? More?