Topics in trees and Catalan numbers

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Math 184A
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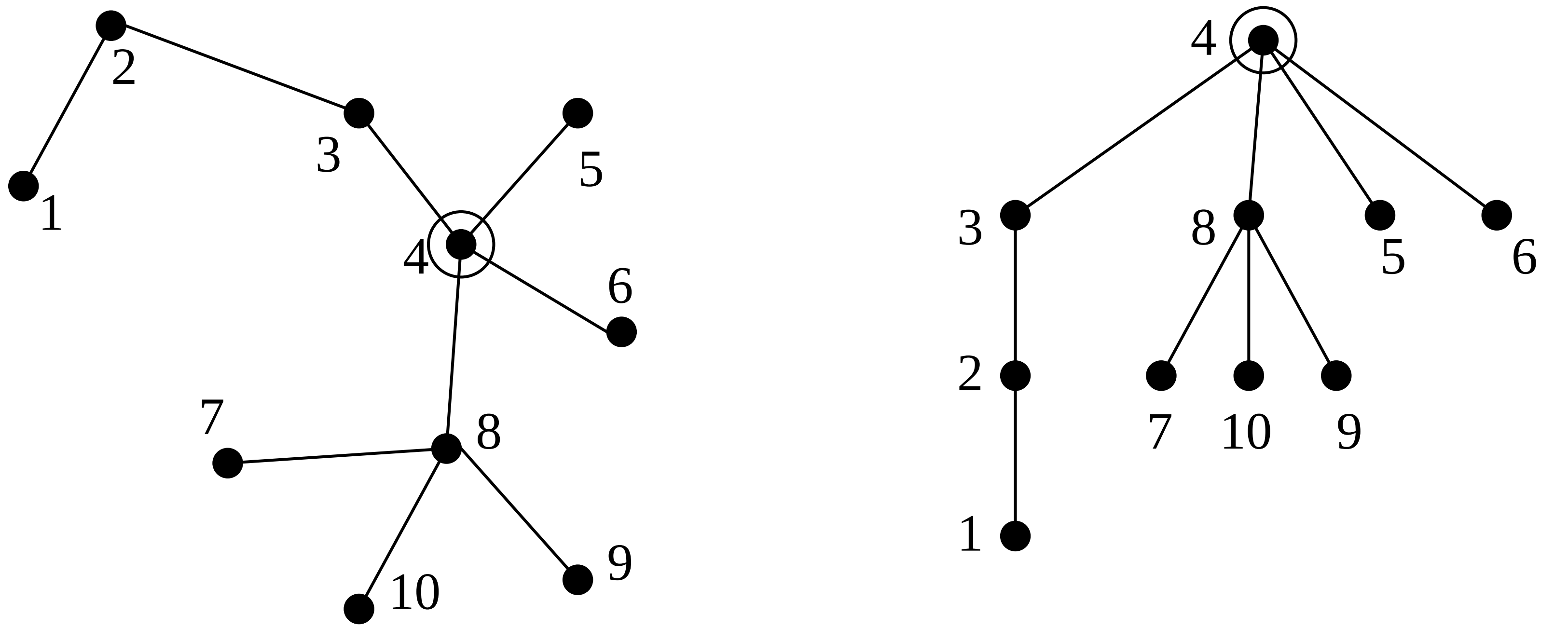
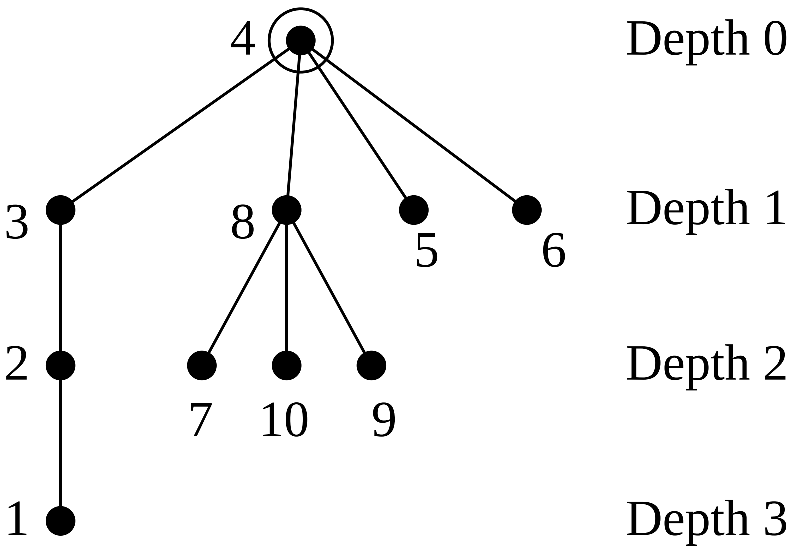
Rooted trees



- A *rooted tree* is a tree with one vertex selected as the *root*.
- It can be drawn in any manner, but it is common to put the root at the top and grow the tree down in levels (or at the left and grow it rightward in levels, etc.).
- The *leaves* are the vertices of degree 0 or 1: 1,5,6,7,10,9.
- The *internal vertices* (or *internal nodes*) are the vertices of degree > 1: 2,3,4,8.

Rooted trees: relationships among vertices



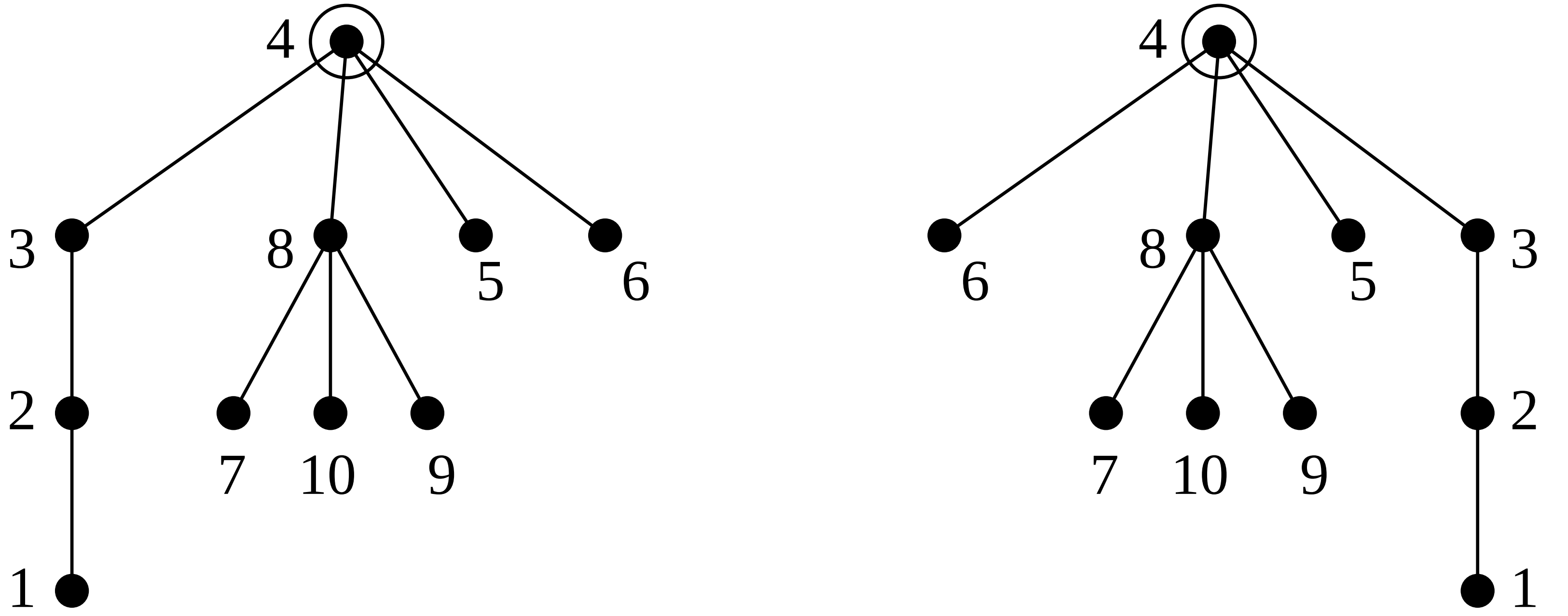
Parents/children

- 4 has *children* 3, 8, 5, 6.
  3, 8, 5, 6 each have *parent* 4.
  3, 8, 5, 6 are *siblings*.
- 8 has parent 4 and children 7, 10, 9.

Depth

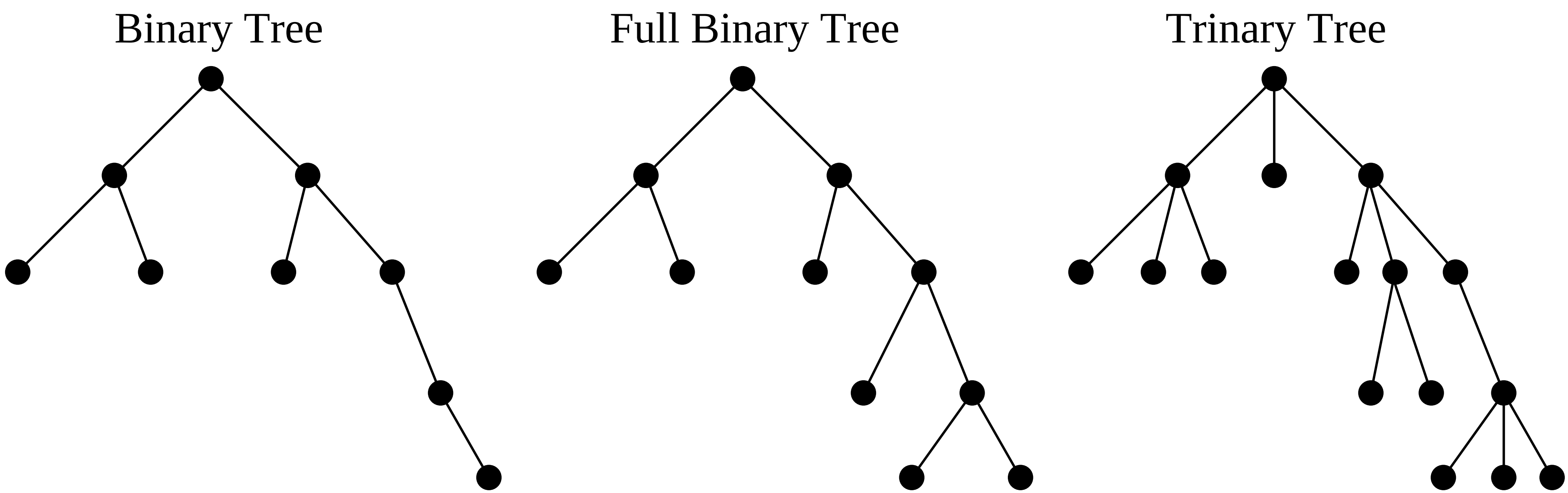
- The *depth* of v is the length of the path from the root to v.
- The *height* of the tree is the maximum depth.

Ordered trees



- An *ordered tree* puts the children of each node into a specific order (pictorially represented as left-to-right).
- The diagrams shown above are the same as *unordered trees*, but are different as *ordered trees*.

Binary trees and more



- In a *k-ary tree*, every vertex has *between* 0 and k children.
- In a *full k-ary tree*, every vertex has *exactly* 0 or k children.
- *Binary*=2-ary, *Trinary*=3-ary, etc.
- *Lemma:* A full binary tree with n leaves has $n-1$ internal nodes, hence $2n-1$ vertices and $2n$ edges in total.

Dyck words

A *Dyck word* (pronounced "Deek") is a string of n 1's and n 2's such that in every prefix, the number of 1's $\geqslant$ the number of 2's.

Example ($n = 5$): 1121211222

Prefix	# 1's	# 2's	#1's $\geqslant$ # 2's
$\emptyset$	0	0	$0 \geqslant 0$
1	1	0	$1 \geqslant 0$
11	2	0	$2 \geqslant 0$
112	2	1	$2 \geqslant 1$
1121	3	1	$3 \geqslant 1$
11212	3	2	$3 \geqslant 2$
$\ldots$			
1121211222	5	5	$5 \geqslant 5$

Dyck words and Catalan numbers

Let W_n be the set of all Dyck words on n 1's and n 2's, and $C_n = |W_n|$ be the number of them.

	W_0	W_1	W_2	W_3
	$\emptyset$	12	1122	111222
			1212	112122
				112212
				121122
				121212
$C_n = \|W_n\|$	1	1	2	5

Catalan numbers $C_n = |W_n|$

- One formula (to be proved later) is

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$$

- $C_3 = \frac{1}{4}\binom{6}{3} = \frac{20}{4} = 5$

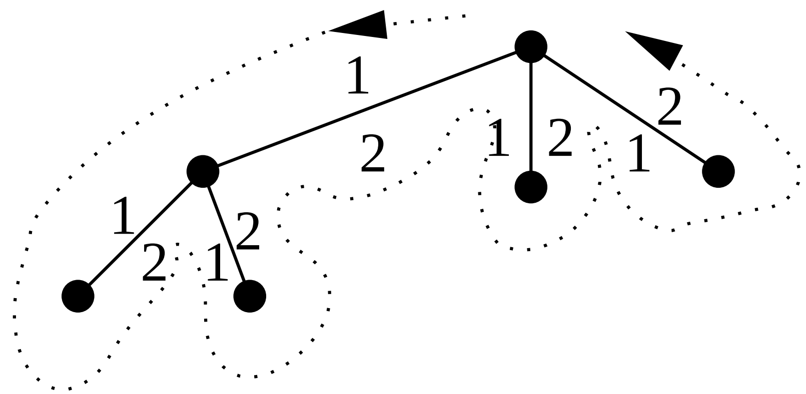
Balanced parentheses

Replacing $1 = ($ and $2 =)$ gives n pairs of balanced parentheses:

	W_0	W_1	W_2	W_3
	$\emptyset$	$12 = ()$	$1122 = (())$	$111222 = ((()))$
			$1212 = ()()$	$112122 = (()())$
				$112212 = (())()$
				$121122 = ()(())$
				$121212 = ()()()$
$C_n = \|W_n\|$	1	1	2	5

Ordered trees

$w = 1121221212$



- Given Dyck word w, form an ordered tree as follows:
  - Draw the root.
  - Read w from left to right.
    For 1, add a new rightmost child to the current vertex and move to it.
    For 2, go up to the parent of the current vertex.
- For any prefix of w with a 1's and b 2's, the depth of the vertex you reach is $a - b \geqslant 0$, so you do not go "above" the root.
  At the end, $a = b = n$ and the depth is $a - b = 0$ (the root).
- Conversely, trace an ordered tree counterclockwise from the root.
  Label each edge 1 going down its left side, and 2 going up its right.
- Thus, W_n is in bijection with ordered trees on n edges (hence $n + 1$ vertices), so C_n counts these too.

Recursion for Catalan numbers

- For $n > 0$, any Dyck word can be uniquely written $u = 1x2y$ where x, y are smaller Dyck words:

  $u = 1\,1212\,2\,112212 \in W_6$ $\quad$ $x = 1212 \in W_2$ $\quad$ $y = 112212 \in W_3$

  $(\,()()\,)\,(())()$ $\quad$ or $x = ()()$ $\quad$ $y = (())()$

- For $u \in W_n$ (with $n > 0$), this decomposition gives $x \in W_i$, $y \in W_{n-1-i}$ where $i = 0, \ldots, n-1$.
- We have $C_0 = 1$ and recursion $\quad C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \quad (n > 0)$.
- $C_0 = 1$
  $C_1 = C_0 C_0 = 1 \cdot 1 = 1$
  $C_2 = C_0 C_1 + C_1 C_0 = 1 \cdot 1 + 1 \cdot 1 = 2$
  $C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5$

Complete binary parenthesization

- What are all ways to parenthesize a product of n letters so that each multiplication is binary?
  E.g., $(a(bc))(de)$ uses only binary multiplications, but $(abc)(de)$ is invalid since abc is a product of three things.
- For a product of n letters, we have $n-1$ binary multiplications:

	$n=1$	$n=2$	$n=3$	$n=4$
	a	ab	$a(bc)$	$((ab)c)d$
			$(ab)c$	$(a(bc))d$
				$a((bc)d)$
				$a(b(cd))$
				$(ab)(cd)$
Count	1	1	2	5

Complete binary parenthesization

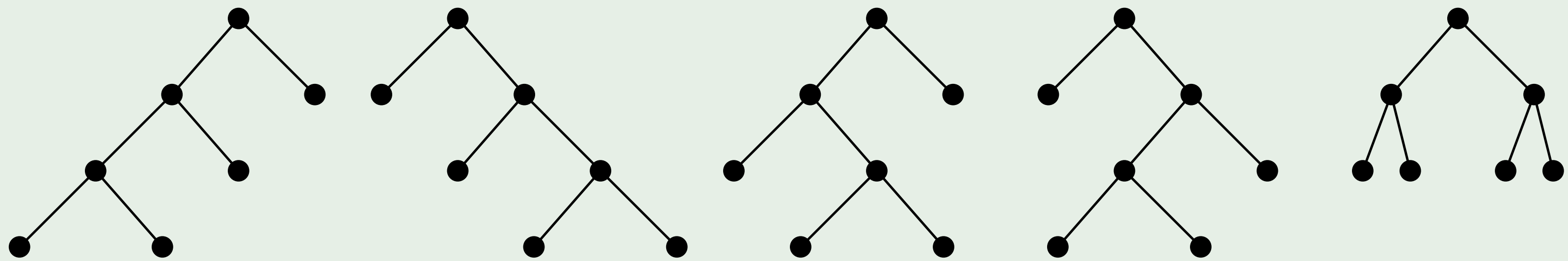
- To parenthesize a product of n letters:
  - Let x be a complete binary parenthesization of the first i letters.
  - Same for y on the other $j = n - i$ letters.
  - Form $(x)(y)$.
    If x or y consists of only one letter, omit the parentheses around it.
- The possible values of i are $i = 1, 2, \ldots, n - 1$.
- Let $B_n = \#$ complete binary parenthesizations of n letters. Then

$$B_1 = 1 \qquad B_n = \sum_{i=1}^{n-1} B_i \, B_{n-i}$$

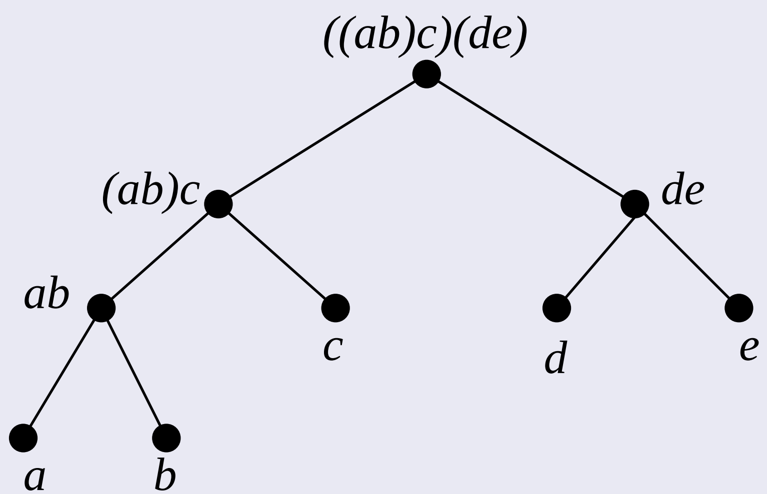
- Similar recursion to Catalan numbers, but n is shifted: $B_n = C_{n-1}$.

Ordered full binary trees

Ordered full binary trees with $n = 4$ leaves



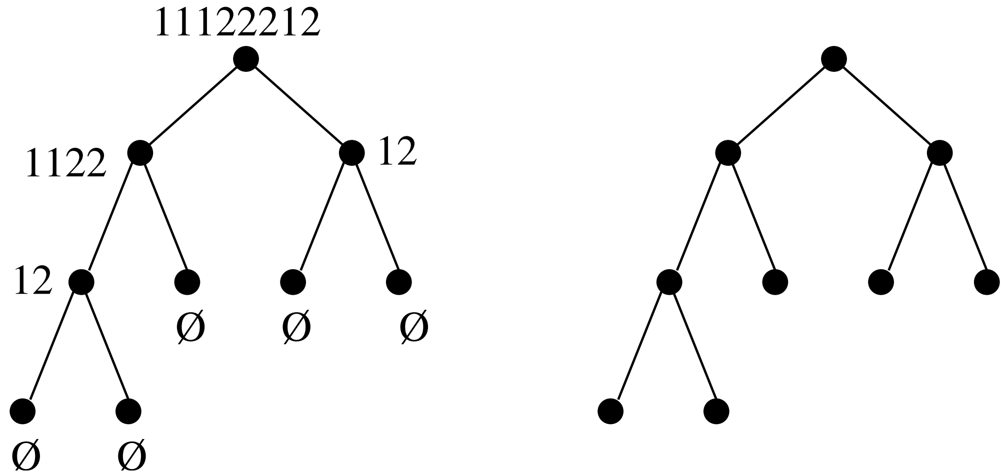
Bijection: ordered full binary trees with n leaves $\leftrightarrow$ complete binary parenthesizations of n factors



Each internal node is labelled by the product of its children's labels, with parentheses inserted for factors on more than one letter.

There are C_{n-1} full binary trees with n leaves.

Bijection: Dyck words $\leftrightarrow$ ordered full binary trees



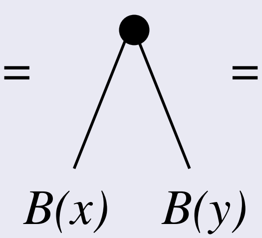
Bijection: Dyck words $W_n \leftrightarrow$ binary tree with $n+1$ leaves

Under the following recursive rules, the word written at the root corresponds to the tree.

Word to tree

B(Ø)= ●

B(1x2y) =

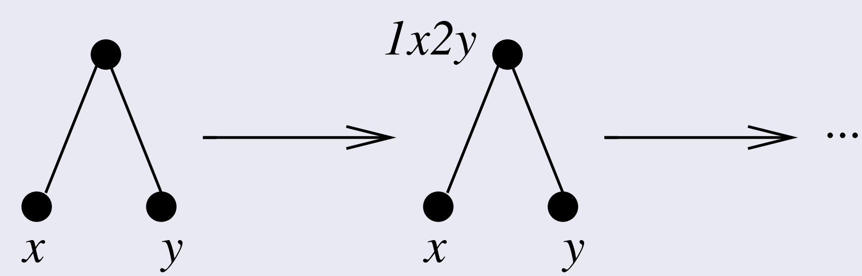


= ...

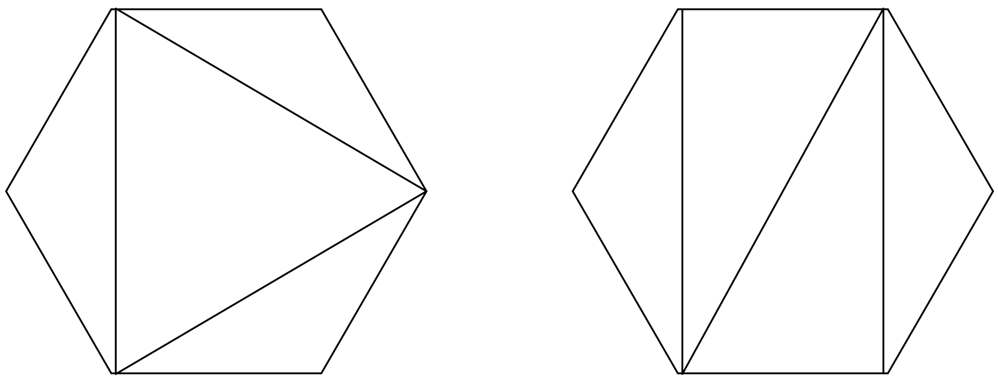
Tree to word

Leaf: ●
Ø

Internal nodes:

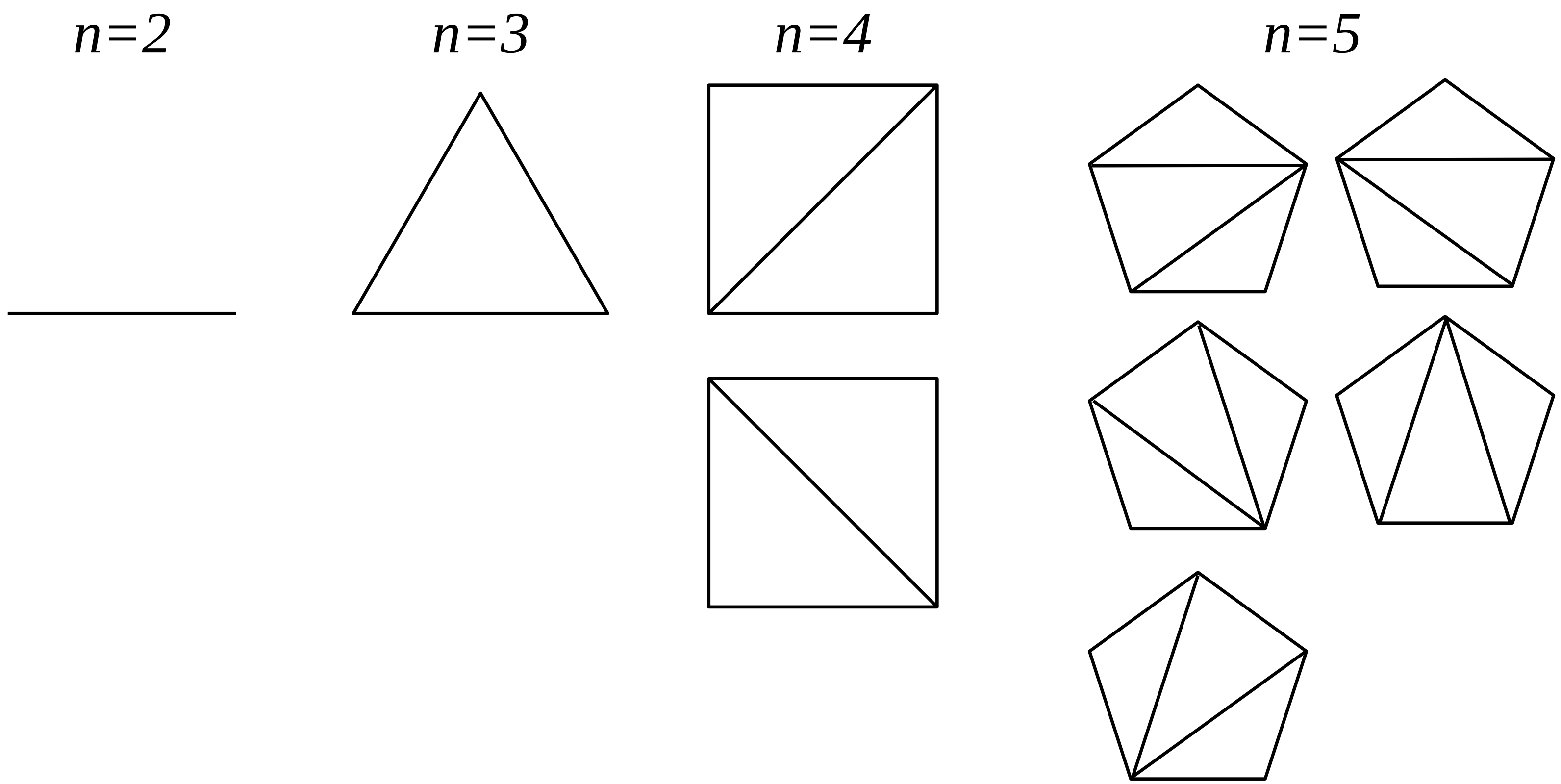


Triangulating regular polygons



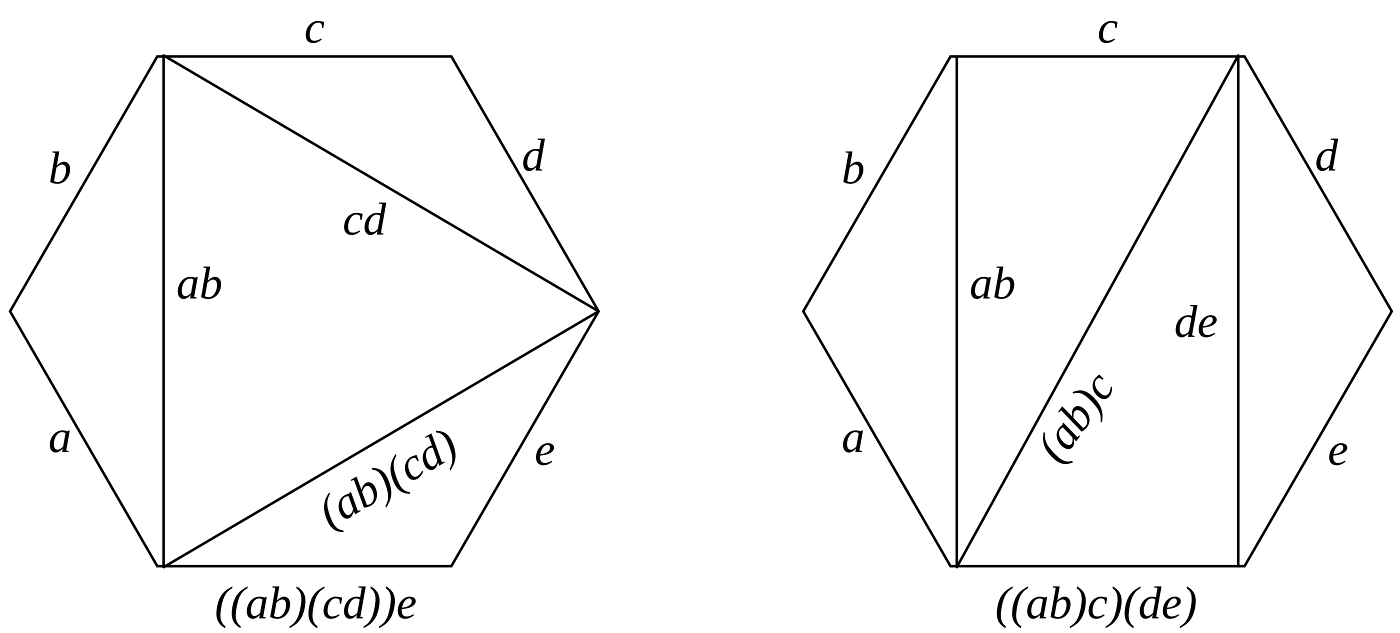
- Draw a regular n-gon, oriented to sit on a horizontal edge on the bottom.
- Draw non-crossing diagonals connecting its vertices, until the whole shape is partitioned into triangular regions.
  In total there are $n-2$ diagonals.
- This is a *triangulation* of the n-gon.

Triangulating regular polygons



The number of triangulations of an n-gon is C_{n-2}.

Triangulating regular polygons



- Draw any triangulation of an n-gon.
- Leave the base empty, and label the other sides by the $n-1$ factors $a, b, c, \ldots$ in clockwise order.
- When you have labels on two sides of a triangle, label the third side by the product of those labels, introducing parentheses to keep the order of multiplications.
- The base is labelled by a complete binary parenthesization on $n-1$ factors. This procedure is reversible.
- The number of triangulations is $C_{(n-1)-1} = C_{n-2}$.

Generating function for Catalan numbers

- Recall that $C_0 = 1$ and
$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \quad (n > 0)$$
- Let $f(t) = \sum_{n=0}^{\infty} C_n t^n$. Then
$$(f(t))^2 = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) t^n$$
$$t(f(t))^2 = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} C_i C_{n-1-i} \right) t^n = \sum_{n=1}^{\infty} C_n t^n = f(t) - 1$$
$$t(f(t))^2 - f(t) + 1 = 0 \quad \text{so} \quad f(t) = \frac{1 \pm \sqrt{1-4t}}{2t} = \frac{1 \pm (1 - 2t + \cdots)}{2t}$$
- Since the series of $f(t)$ starts at $1t^0$, the solution is $f(t) = \frac{1-\sqrt{1-4t}}{2t}$.

Generating function for Catalan numbers

- On homework 2, you computed the binomial series for $\sqrt{1+x}$:

$$(1+x)^{1/2} = \cdots = 1 - 2\sum_{n=1}^{\infty} \frac{(-1/4)^n (2n-2)!}{n!(n-1)!} x^n$$

- Set $x = -4t$:

$$\sqrt{1-4t} = 1 - 2\sum_{n=1}^{\infty} \frac{(-1/4)^n (2n-2)!}{n!(n-1)!} (-4t)^n$$

$$= 1 - 2\sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!} t^n$$

$$f(t) = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!} t^{n-1} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)!n!} t^n$$

- Since $f(t) = \sum_{n=0}^{\infty} C_n t^n$, we proved $C_n = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1}\binom{2n}{n}$.

- The *weight* of a word $u \in W_n$ is $w(u) = n$.
- When $n > 0$, decompose $u = 1x2y$ where x, y are Dyck words. Then
$$w(u) = w(12) + w(x) + w(y) = 1 + w(x) + w(y)$$

- The product formula gives the generating function by weight for all Dyck words with $n \geqslant 1$ is $t \cdot f(t) \cdot f(t)$.
- This generating function is also $f(t) - 1$.
- Thus, $t(f(t))^2 = f(t) - 1$, giving the same equation as before.

Summary

We looked at many classes of objects that are counted by Catalan numbers, and we established bijections between them:

- C_n Dyck words on n 1's and n 2's.
- C_n ways to form n pairs of balanced parentheses.
- C_n ordered trees with n edges (hence $n+1$ vertices).
- C_{n-1} complete binary parenthesizations of a product $x_1 x_2 \ldots x_n$.
- C_{n-1} ordered full binary trees with n leaves.
- C_{n-2} triangulations of an n-gon.

There are 100's more classes of objects people have found that are counted by Catalan numbers.