PIGEON-HOLE PRINCIPLE

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The pigeon-hole principle is very simple, but it can be used to prove a remarkably wide range of problems, some quite difficult. It says the following: if you place $n+1$ pigeons in n holes, there the must exist at least one hole with at least two pigeons in it. This can be generalized as follows: if you place $kn + 1$ pigeons in n holes, there must exist at least one hole with at least $k + 1$ pigeons.

The following problems are taken from K. Rosen "Discrete Mathematics and Applications".

1. Show that there are at least six people in California (population 36 million) with the same three initials who were born on the same day of the year (but not necessarily in the same year). Assume that everyone has three initials.

2. During the month of June, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team plays exactly 14 games.

3. Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

4. Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

5. (Ramsey type problem) Assume that in a group of 6 people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

6. Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

7. Show that if there are 101 people of different heights standing in a line, it is possible to find 11 people in the order they are standing in the line with heights that are either increasing or decreasing.

The following problems were collected by M. Bestvina and G. Savin for Utah's Putnam team training.

Number Theory.

(1) Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \cdots, 100$. Prove that there must be two distinct integers in A whose sum is 104. Hint: Consider "holes" $\{1\}, \{4, 96\}, \{7, 93\}, \cdots$.

(2) A *lattice point* in the plane is a point (x, y) such that both x and y are integers. Find the smallest number n such that given n lattice points in the plane, there exist two whose midpoint is also a lattice point. Hint: Consider as "holes" coordinates mod 2.

(3) Show that there exist two different powers of 3 whose difference is divisible by 2005.

(4) Show that some multiple of 2005 is a number all of whose digits are 0's and 1's. Hint: Consider the sequence $1, 11, 111, \cdots$.

(5) Show that there exists a power of 3 that ends with 001.

(6) Given any sequence of n integers, some consecutive subsequence has the property that the sum is a multiple of n.

(7) Given any 1000 integers, some two differ by, or sum to, a multiple of 1997. Hint: "Holes" are $\{0\}, \{1, 1996\}, \{2, 1995\}, \cdots$ (mod 1997).

(8) If $n + 1$ of the integers from $\{1, 2, \cdots, 2n\}$ are selected, then some two of the selected integers have the property that one divides the other. Hint: Multiply by the largest power of 2 you can and still stay $\leq 2n$.

(9) Out of the set of integers $1, 2, \cdots, 100$ you are given 10 different integers. From this set show that you can always find two disjoint nonempty subsets S and T with equal sums. Hint: How many subsets can you form?

(10) Let α be an irrational number and let S be the set of all real numbers of the form $a + b\alpha$ where a, b are integers. Show that S is *dense* in the set of real numbers, i.e. any interval (u, v) with $u < v$ contains elements of S. Hint: Fix an integer $Q > 0$ and consider the holes $[0, 1/Q), [1/Q, 2/Q), \cdots, [(Q-1)/Q)$. Send an integer b to a hole according to the value of $\{b\alpha\}$ (the fractional part of $b\alpha$).

(11) Prove that there is some integral power of 2 that begins $2005 \cdots$. Hint: Show first that $\log 2$ is irrational.

(12) **(Erdös-Szekeres theorem)** Every sequence of $mn + 1$ distinct real numbers has either an increasing subsequence of length $m + 1$ or a decreasing subsequence of length $n + 1$. Hint: Put a member a of the sequence into the hole labeled r if the longest increasing subsequence starting at a is r.

(13) **(Dirichlet's theorem)** Let α be an irrational number. Then there are infinitely many integer pairs (h, k) with $k > 0$ such that

$$\left| \alpha - \frac{h}{k} \right| < \frac{1}{k^2}$$

Hint: Just like #10.

(14) (B2 2006) Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n + 1}.$$

Hint: For every $i = 1, \ldots, n$ define $y_i = x_1 + x_2 + \cdots + x_i$ and $z_i = y_i - [y_i]$. Note that z_i is contained in $[0, 1)$. Use $[0, \frac{1}{n+1}), \ldots, [\frac{n}{n+1}, 1)$ as "holes".

(15) **(Proizvolov's Identity)** In any partition of $1, 2, \cdots, 2n$ into decreasing and increasing sequences of n numbers each, the sum of the absolute values of the differences of the corresponding terms is always n^2. Hint: If a_i, b_i are corresponding terms, show that one of the numbers is $\leq n$ and the other is $\geq n + 1$.

Geometry.

(16) Prove that among any 5 points selected inside an equilateral triangle with sidelength 1, there always exists a pair at distance no more than $1/2$. Hint: Subdivide into 4 copies of a scaled down triangle.

(17) Given any 6 points inside a circle of radius 1, some two of the points are within 1 of each other.

(18) Mark the centers of all squares of an 8×8 chess board. Is it possible to cut the board with 13 straight lines (none passing through a marked point) so that every piece has at most one marked point. Hint: Focus on the squares along the edge of the board.

(19) Given a planar set of 25 points such that among any 3 of them there exists a pair at distance < 1, prove that there exists a circle of radius 1 centered at one of the points that contains at least 13 of the points. Hint: Focus on two points at distance ≥ 1.

(20) Suppose each point in the plane is colored red or blue. Show that some rectangle has its vertices all the same color. Hint: Fix 3 horizontal lines and look for a rectangle with a pair of sides on two of the lines.

(21) Given any $mn + 1$ intervals, there exist $n + 1$ pairwise disjoint intervals or $m + 1$ intervals with a nonempty intersection.

(22) A polygon in the plane has area > 1. Show that it contains two distinct points (x_1, y_1) and (x_2, y_2) that differ by (a, b) where a and b are integers.

Combinatorics.

(23) Given any 10 4-element subsets of $\{1, 2, \cdots, 11\}$, some two of the subsets intersect in at least 2 elements.

(24) Show that if more than half of the subsets of $\{1, 2, \cdots, n\}$ are selected, then some two of the selected subsets have the property that one is contained in the other. Hint: A typical hole is $\{A, A \cup \{n\}\}$.

(25) Show that if more than half of the subsets of $\{1, 2, \cdots, n\}$ are selected, then some two of the selected subsets are disjoint.

(26) Show that in any group of 6 people there are either 3 mutual friends or 3 mutual strangers.

(27) In a follow-up to the previous question, show that there is some n so that in any group of n people there are either 4 mutual friends or 4 mutual strangers.

(28) At a round table with n seats there are at least $(k - 1)n/k$ people seated. Show that there exist k consecutive chairs that are occupied. Hint: If $k|n$ it should be easy. In general arrange this by passing to a multiple of n and "unwrapping".

(29) Let M be a 14×14 matrix with all entries 0 or 1. If M has 58 entries equal to 1, then there are two rows and two columns so that all 4 entries in their intersection are 1's. Hint: This is similar to #19.

(30) The squares of an $n \times n$ chess board are labeled arbitrarily with numbers 1 through n^2. Prove that there are two adjacent squares whose labels differ by at least n. Hint: Put the labels one at a time starting with $1, 2, \cdots$. Stop when a row or a column gets filled up. Now argue that there are n pairwise disjoint 2×1 subrectangles with one square filled and one empty.

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