

# Trax, Trains and Trolleys

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## Part 1: Easy Counting

There are elementary counting principles that we want to make sure everyone is comfortable with. For example,

- (1) If your little sister has 4 different skirts and 7 different shirts, then how many different skirt-shirt outfits can she assemble, assuming nothing clashes?

*The answer  $4*7=28$ , because you can make a rectangular array of all the possibilities. For example, in the table below, the first number represents the skirt choice, and the second number represents the shirt choice. This array has 4 rows and 7 columns.*

|       |       |       |       |       |       |       |
|-------|-------|-------|-------|-------|-------|-------|
| (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) | (1,7) |
| (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) | (2,7) |
| (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | (3,6) | (3,7) |
| (4,1) | (4,2) | (4,3) | (4,4) | (4,5) | (4,6) | (4,7) |

*Thus, if you have  $m$  choices for one operation, and  $n$  choices for a successive operation, then the number of ways in which you can do the first followed by the second, is  $m*n$ .*

- (2) What if your sister has 10 different pairs of socks? How many skirt-shirt-sock selections are possible?

*The answer is  $4*7*10 = 28*10 = 280$ , since you already knew she could make 28 skirt-shirt choices, and for each of those she has 10 sock possibilities.*

- (3) Suppose that there are 40 people in this room, and that everyone shakes hands exactly once with everyone else. How many handshakes occur?

*Pretend you are the social director. You have 40 choices for the first shaker, and then 39 choices for the second, so  $40*39$  is your first guess. But wait, this causes duplication, since you could pick George first and then Sammie, as well as Sammie first and then George. Thus your final answer is  $(40*39)/2$ .*

O.K., so those problems were too easy. Can you do these?

- (4) What if we did triple handshakes (groups of 3) instead of the usual kind?
- (5) In a 52-card deck in which there are 4 suits (hearts, diamonds, clubs, spades) and 13 denominations (2,3 ...9, 10, Jack, Queen, King, Ace), how many 5-card hands are there consisting of "three of a kind," 3 cards of the same denomination (e.g. 3 Jacks), and two other cards which don't match each other or the 3 cards.
- (6) We need to take a group picture of Alice, Ben, Charlie, Doris and Eileen. We will stand all 5 kids in a line, except we cannot put Charlie next to Ben because if we do they'll start fighting. And Alice can't stand next to Eileen because they'll start gossiping. How many ways can we arrange the kids? (Even easy counting can be difficult.)

**Part 2: Combinations, and Pascal's Triangle**

As part of many counting problems you need to figure out how many ways you can choose  $k$  objects from a collection of  $n$ , in which the order of selection is irrelevant. We call this number “the combinations of  $n$  objects chosen  $k$  at a time,” or “ $n$  choose  $k$ ” for short, and write it as  $\binom{n}{k}$ . For example, in problem (4) you computed  $\binom{40}{3}$ .

We define  $\binom{n}{0}=1$ , which also equals  $\binom{n}{n}$ .

For  $1 \leq k \leq n$ , we can find the formula for  $\binom{n}{k}$ , by careful counting. We create our combination by successively picking objects: There are  $n$  choices for the first element, then  $(n-1)$ , then  $(n-2)$ , and finally  $(n-k+1)$  choices for the last element in our collection. By the multiplication principle there are  $n*(n-1)*(n-2)*...*(n-k+1)$  ways of making these choices. Of course, many choices yield the same combination because the order of our collection can be permuted (mixed up). In fact, we can permute a given combination in exactly  $k*(k-1)*...3*2*1 = k!$  (“ $k$  factorial”) ways. (Why?) Thus by enumerating all selections of  $k$  objects in which the order counts, we will list each combination exactly  $k!$  times. So we get the actual number of combinations by dividing:

$$\binom{n}{k} = \frac{n*(n-1)*...*(n-k+1)}{k*(k-1)*...*3*2*1} = \frac{n!}{(n-k)!*k!}$$

Notice the last expression looks shorter than the one in the middle, but would take longer to compute if you expanded it without canceling terms. Also, if we define  $0!=1$ , then the last expression is also O.K. when  $k=0$  or  $n$ .

It's interesting to make a table in which you list the combination coefficients in a triangular array. Here are the first 5 rows of what is actually an infinite triangle:

|                |                |                |                |                |                |                |                |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
|                |                |                | $\binom{0}{0}$ |                |                |                |                |
|                |                |                | $\binom{1}{0}$ |                | $\binom{1}{1}$ |                |                |
|                |                | $\binom{2}{0}$ |                | $\binom{2}{1}$ |                | $\binom{2}{2}$ |                |
|                | $\binom{3}{0}$ |                | $\binom{3}{1}$ |                | $\binom{3}{2}$ | $\binom{3}{3}$ |                |
| $\binom{4}{0}$ |                | $\binom{4}{1}$ |                | $\binom{4}{2}$ |                | $\binom{4}{3}$ | $\binom{4}{4}$ |

This array is called Pascal's triangle, and when you start computing the entries you might notice an interesting pattern:

Pascal's Triangle

|   |   |   |   |   |    |    |    |    |    |    |    |    |    |   |   |   |   |
|---|---|---|---|---|----|----|----|----|----|----|----|----|----|---|---|---|---|
|   |   |   |   |   |    |    | 1  |    |    |    |    |    |    |   |   |   |   |
|   |   |   |   |   |    | 1  |    | 1  |    |    |    |    |    |   |   |   |   |
|   |   |   |   |   | 1  |    | 2  |    | 1  |    |    |    |    |   |   |   |   |
|   |   |   |   | 1 |    | 3  |    | 3  |    | 1  |    |    |    |   |   |   |   |
|   |   |   | 1 |   | 4  |    | 6  |    | 4  |    | 1  |    |    |   |   |   |   |
|   |   | 1 |   | 5 |    | 10 |    | 10 |    | 5  |    | 1  |    |   |   |   |   |
|   | 1 |   | 6 |   | 15 |    | 20 |    | 15 |    | 6  |    | 1  |   |   |   |   |
| 1 |   | 1 |   | 7 |    | 21 |    | 35 |    | 35 |    | 21 |    | 7 |   | 1 |   |
|   | 1 |   | 8 |   | 28 |    | 56 |    | 70 |    | 56 |    | 28 |   | 8 |   | 1 |

Do you see the pattern that allows you to get any row (after the first one) from the row above it: The entry in row  $(n+1)$  is the sum of the two entries diagonally above it. Is this an accident for the first 9 rows, or can you:

(7) Show that for  $1 \leq k \leq n-1$ ,

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

Notice, by an application of mathematical induction so simple that we don't usually mention it, this identity implies that every row of Pascal's triangle yields the correct combination coefficients:

- (I) Row 1 is correct
- (II) If row  $n$  is correct, then the identity (7) shows that row  $n+1$  is correctly recording the combination coefficients, so row  $n+1$  is correct.

Thus by induction, all rows are correct!

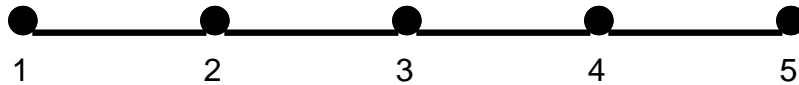
**Part 3: Trax, trolleys and trains** (This section is motivated by Chapter 6, “Passage to Pentagonia,” in Ian Stewart’s wonderful book, “Another Fine Math You’ve Got Me Into.”)

Imagine a lazy Friday afternoon, and you’re hanging out with your mathy friends in downtown Salt Lake City. Your Mom or Dad didn’t give you any allowance this week so you’re short on cash. Hey, here’s a fun game, let’s ride TRAX in the free-fare zone! (See map on next page.) As it turns out there are 5 stations in this zone, and they run in a straight line. They are: Delta Center, Temple Square, City Center, Gallivan Plaza and Courthouse. You’ve been talking about counting in Math Circle, and you ask your friends the following question:

(7) Suppose I start at City Center, the middle station. I will count as a “leg” any TRAX excursion from a station to an adjacent station. Thus going from City Center to Gallivan Plaza is a leg, from Gallivan to Courthouse is a leg, and so on. I wonder how many trips I can make which start and end at City Center, consist of exactly 10 legs, and stay in the free fare zone?

Let’s make life easier and number our stations from 1 to 5, with City Center equal to station 3. After you do this, you could write down all possible trips, but this might take awhile. For example, here’s a possibility:

$$3 \rightarrow 4 \rightarrow 5 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 3$$

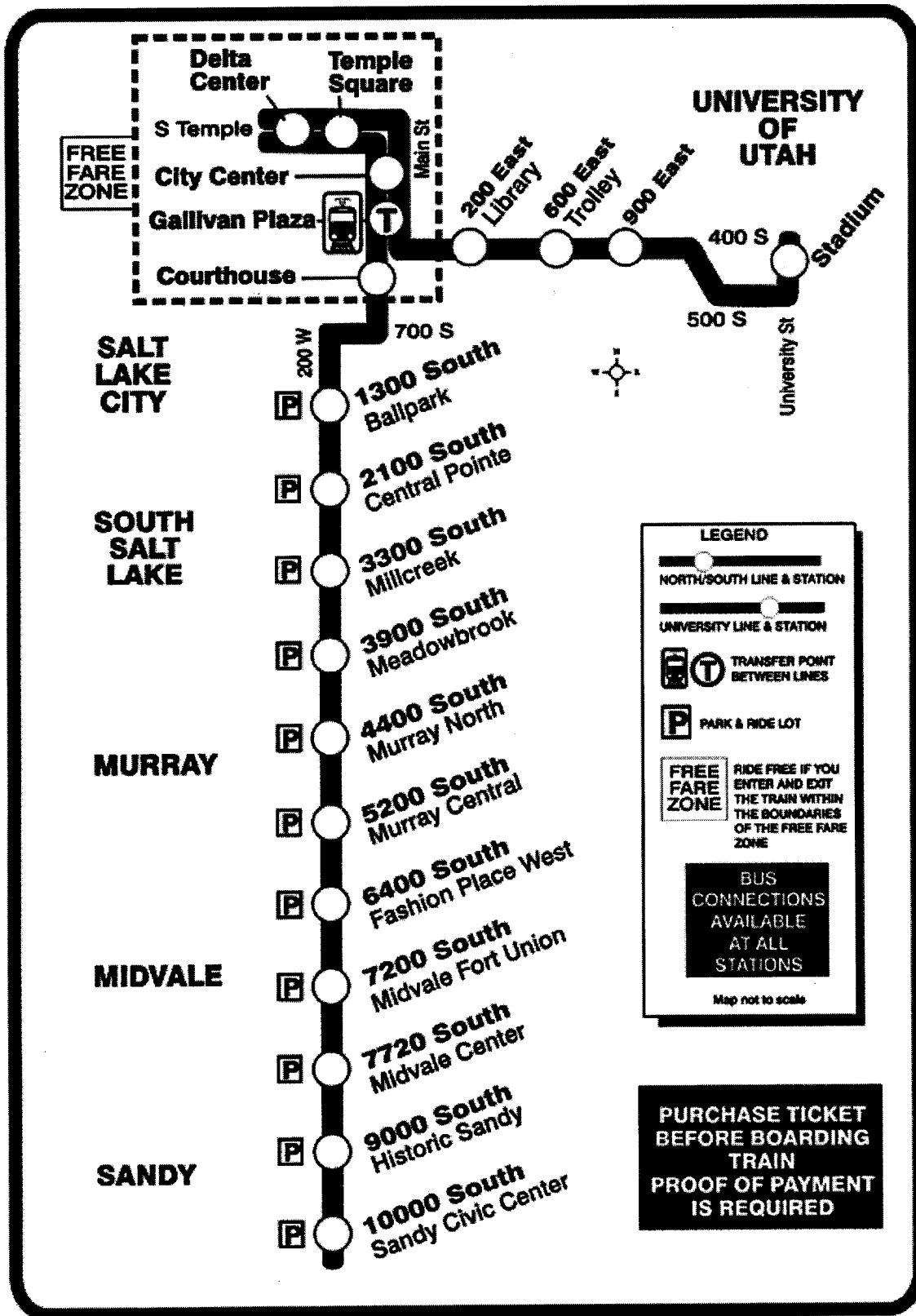


Maybe we can recursively figure out possible trips, kind of like how we made Pascal’s triangle. In other words, there is one trip of length zero starting at City Center, two trips of length 1 (one going to 2, the other to 4). If we know of trips of length  $n$  we can figure out the ones of length  $n+1$ :

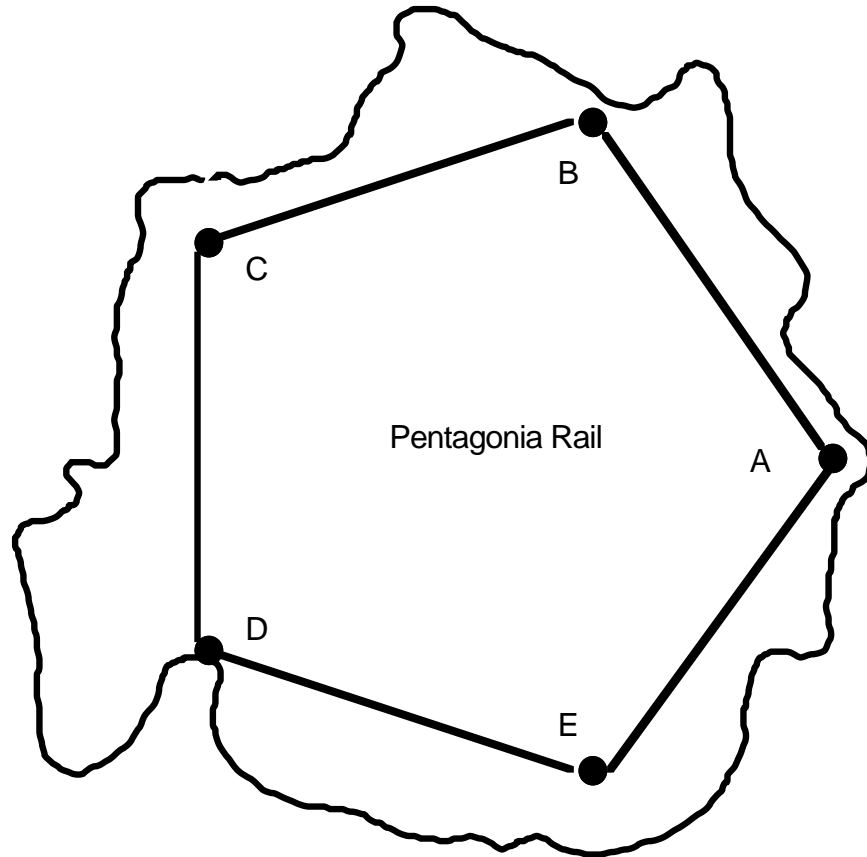
|    |    |     |    |    |
|----|----|-----|----|----|
| 0  | 0  | 1   | 0  | 0  |
| 0  | 1  | 0   | 1  | 0  |
| 1  | 0  | 2   | 0  | 1  |
| 0  | 3  | 0   | 3  | 0  |
| 3  | 0  | 6   | 0  | 3  |
| 0  | 9  | 0   | 9  | 0  |
| 9  | 0  | 18  | 0  | 9  |
| 0  | 27 | 0   | 27 | 0  |
| 27 | 0  | 54  | 0  | 27 |
| 0  | 81 | 0   | 81 | 0  |
| 81 | 0  | 162 | 0  | 81 |
|    |    |     |    |    |

So, the answer is that there are 162 possible trips.

(8) How many trips of length 100 legs start and end at City Center? Could you prove it?



Well you worked that out easily! But now, your friend Abramo says, “You know, this kind of reminds me of my trip to Pentagonia last summer – an island in Southern Argentina. There are 5 main cities in Pentagonia, and their names are Abbindolare, Bancarotta, Canzonatura, Dappoco, and Esoso, but let’s call them A,B,C,D,E. (You could ask Renzo what these names mean in Italian.) Here’s how the railroads are laid out on that island paradise.

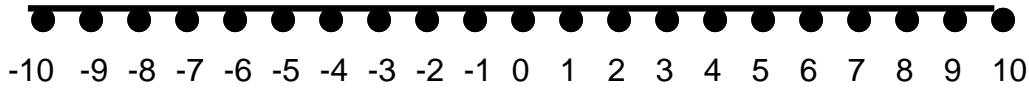


(9) How many different train trips consisting of 10 legs, starting and ending at Canzonatura, are there? Can you express your answer in terms of combinations? We could make a table like we did for TRAX. This is what we’d get:

|     |     |     |     |     |
|-----|-----|-----|-----|-----|
| 0   | 0   | 1   | 0   | 0   |
| 0   | 1   | 0   | 1   | 0   |
| 1   | 0   | 2   | 0   | 1   |
| 1   | 3   | 0   | 3   | 1   |
| 4   | 1   | 6   | 1   | 4   |
| 5   | 10  | 2   | 10  | 5   |
| 15  | 7   | 20  | 7   | 15  |
| 22  | 35  | 14  | 35  | 22  |
| 57  | 36  | 70  | 36  | 57  |
| 93  | 127 | 72  | 127 | 93  |
| 220 | 165 | 254 | 165 | 220 |
|     |     |     |     |     |

So, the answer is that there are 254 possible trips.

We can “unwrap” *Pentagonia* to express the answer in terms of combinations. By unwrap, I mean we will number *C* as zero, and replace *Pentagonia Rail* with a number line:



Then if we end up at 0, 5, -5, 10, or -10 on the unwrapped line, we would've ended up at *Canzonatura* on *Pentagonia*! To end up at 0 I must move right 5 times, and left 5 times. The number of ways I can do this is the combinations of 10 things chosen 5 at a time. I can't end up at 5 or -5 if I do 10 moves!!!! (After an even number of moves you're always at an even number.) However I can end up at 10 if I always move right, or at -10 if I always move left. Thus the number of 10-leg trips ending up at *Conaonatura* is

$$\binom{10}{5} + 2 = \frac{10 * 9 * 8 * 7 * 6}{5 * 4 * 3 * 2 * 1} + 2 = 252 + 2 = 254$$

#### Part 4: Incidence Matrices

Your other friend, *Clarisse*, is trying to think of a way to write a procedure for computing train trips, for any train system. She's good at computing. She decides to keep track of all trips from station *i* to station *j* in a matrix. She starts with trips of length 1. For example, if we number our *Pentagonian* cities from 1 to 5, with *Abbindolare* = 1 and *Esoso* = 5, the matrix is given by:

$$P := \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

If you look down the third column we see that we can take a trip of length 1 from 3 → 2, and also one from 3 → 4. This is reflected by the 1 which appears in entries (2,3) and (4,3) of the matrix. (Remember, you say where you are in a matrix by specifying (row, column).) There are 0's in the (1,3),(3,3),(5,3) entries because there is no rail line between those cities. In fact this matrix is called **the incidence matrix** for the graph of *Pentagonia rail*, because there is a 1 in the *ij* location if and only if there is an edge connecting vertex *i* to vertex *j*. If there is no edge, then the *ij*-entry is 0.

Now *Clarisse* isn't sure how to make the table for trips of length 2. Playing with her calculator keys she accidentally starts to take powers of *P* (multiplying *P* by itself

over and over again.) Remember the weird way that matrix multiplication was defined in your algebra class? Here's what she gets:

$$P^2 = \begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 3 & 1 & 1 & 3 \\ 3 & 0 & 3 & 1 & 1 \\ 1 & 3 & 0 & 3 & 1 \\ 1 & 1 & 3 & 0 & 3 \\ 3 & 1 & 1 & 3 & 0 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 6 & 1 & 4 & 4 & 1 \\ 1 & 6 & 1 & 4 & 4 \\ 4 & 1 & 6 & 1 & 4 \\ 4 & 4 & 1 & 6 & 1 \\ 1 & 4 & 4 & 1 & 6 \end{bmatrix}$$

$$P^{10} = \begin{bmatrix} 254 & 165 & 220 & 220 & 165 \\ 165 & 254 & 165 & 220 & 220 \\ 220 & 165 & 254 & 165 & 220 \\ 220 & 220 & 165 & 254 & 165 \\ 165 & 220 & 220 & 165 & 254 \end{bmatrix}$$

Check out those 3<sup>rd</sup> columns, and compare them to your Pentagonia Rail Table. This can't be an accident?

(Could you prove that the  $ij$  entry in the matrix  $P^n$  gives you the number of trips from vertex  $j$  to vertex  $i$  (or vice versa), with  $n$  legs. This is a true theorem.)



**Challenge Problem:**

Here is a map of the Trolley system in Peacity. How many 100-leg trips are possible starting and ending at Nirvana? Can you find (and justify) a formula for the number of such trips consisting of  $n$  legs?

