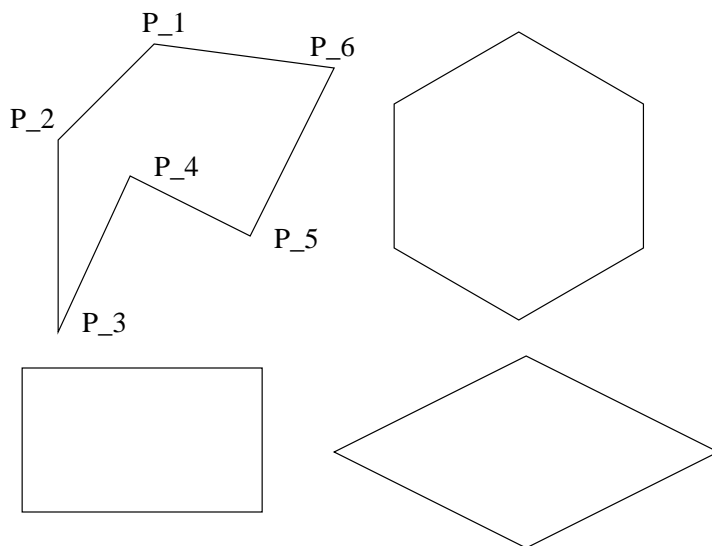


What is symmetry?

Polygons

By a *polygon* we mean a sequence of m line segments (“sides”) $P_1P_2, P_2P_3, \dots, P_{m-1}P_m, P_mP_1$ joining m distinct points P_1, \dots, P_m (“vertices”). We will call such a figure an “ m -gon”. We will insist that the sides do not cross one another. And, usually, we will insist that the points P_1, \dots, P_m lie in a common plane. In this case, the polygon is called *planar*. A planar polygon is called *convex* if it encloses a convex region in the plane— i.e. every line segment between two points in this region lies entirely in the region. A polygon is said to be *regular* if every side has the same length (“equilateral”) and if every interior angle has the same measure (“equiangular”).



Question 1. Which of the above polygons is planar, convex, equiangular, equilateral, and/or regular? Is it possible to be equiangular and not equilateral? Vice-versa? For any number of sides?

Question 2. *Suppose each of the edges of a regular m -gon is assigned a length of one. What is its perimeter? What is its area?*

Symmetries

We probably have a good feel for the notion of ‘symmetry’. Like art, we know it when we see it. On the other hand, it is difficult to say what is art and what is not. Our working definition of (mathematical) symmetry will be the following:

Say that a figure has *symmetry* if there is a distance preserving transformation of space which preserves the figure. Such a transformation is called an *isometry*. Now, you may contend that this is a complete definition of ‘symmetry’. On the other hand, you should agree that this definition captures the spirit of many familiar symmetries.

Question 3. *Which of the following transformations of the plane preserve distance?*

(1) $(x, y) \mapsto (2x, 2y)$

(2) $(x, y) \mapsto (-y, x)$

(3) $(x, y) \mapsto (2 - x, y)$

- (4) $(x, y) \mapsto (x \cos(\pi/3) + y \sin(\pi/3), -x \sin(\pi/3) + y \cos(\pi/3))$
 (Hint: Start by sketching where the points $(1, 0)$ and $(0, 1)$ are mapped.)

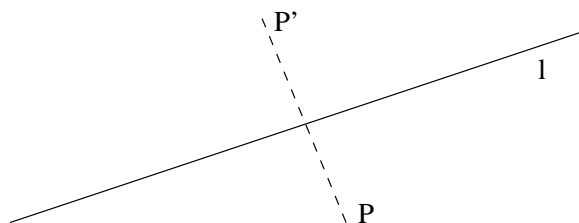
Challenge 1. In the above transformations, many of these can be described by a matrix. Namely, there is a 2×2 matrix A so that $(x, y) \mapsto (x, y)A$. Find some of these. What are their determinants?

Reflections

One of the most elementary distance preserving transformations is a *reflection*. We may describe a reflection in the plane in terms of an invariant line. If ℓ is a line in the plane, then we may describe the transformation ‘reflection in ℓ ’ as follows:

If P lies on ℓ , then P is mapped to itself. If P does not lie on ℓ , then P is mapped to the unique point P' satisfying

- PP' is perpendicular (orthogonal) to ℓ , and
- P and P' are equidistant from ℓ



Question 4. *Were any of the previous transformations reflections?*

A figure in the plane which is preserved (i.e. mapped to itself) by a reflection in a line is said to have a ‘line of symmetry’.

Question 5. *How many lines of symmetry does a regular m -gon have?*

The notion of reflection makes sense in any dimension. In one dimension, for instance, we can define the transformation ‘reflection in a point’. In three dimension, we can define ‘reflection in a plane’.

Question 6. *How many planes of symmetry does the cube have?*

Rotations

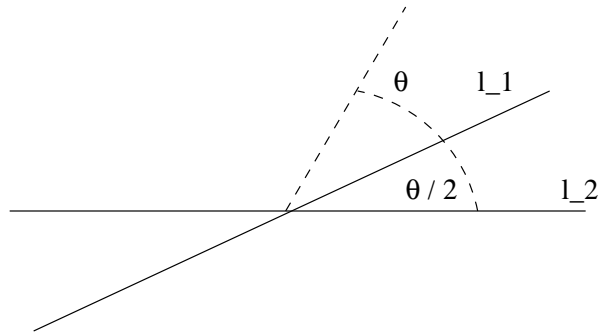
Another common isometry is rotation. In two dimensions, we may rotate about a central point. In three dimensions, we may rotate about an axis. We will refer to these as ‘points’ and ‘axes’ of ‘rotational symmetry’.

Question 7. *How many points of rotational symmetry does a regular m -gon have? How many different rotations preserve this m -gon?*

Question 8. *How many axes of rotational symmetry does the cube have? How many different rotations preserve the cube?*

There is a simple relationship between reflections and rotations. Try to convince yourself of the following fact:

In the plane, rotation by angle θ about the origin is the same transformation as the composite of two reflections, namely compose r_1 and r_2 where each r_i is a reflection in a line ℓ_i passing through the origin and the lines ℓ_1 and ℓ_2 form an angle measuring $\theta/2$.



Question 9. *Does it matter in which order you compose the reflections?*

Challenge 2. *Formulate and prove the corresponding statement for reflections and rotations in 3-space. What about n -space?*

Question 10. *Not every rotational symmetry of a plane figure is the composite of two reflections. Give an example of such a figure. (Hint: draw a figure which has no lines of symmetry, but which has a point of rotational symmetry.)*

Symmetry Groups

Suppose we have a figure X in the plane (or in 3-space). A natural question to ask is ‘what are all the isometries of the plane (or 3-space) which preserve the figure X . This collection of symmetries (why should these be called symmetries?) of X is called the ‘symmetry group of X ’. This seems to be a reasonable measure of how symmetric the figure X really is.

For us, we will ask simpler (yet, in fact, equivalent) question of how many different isometries can be obtained by composing elementary symmetries (reflections and rotations) of a plane figure.

Question 11. *How large is the symmetry group of the m -gon? How large is the symmetry group of a rectangle? A triangle? A circle?*

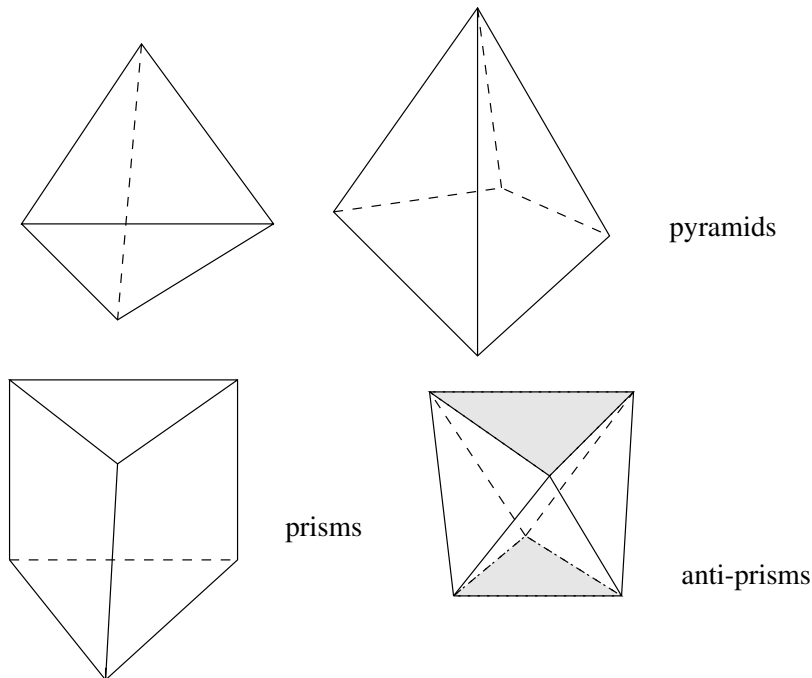
Question 12. *What are the most symmetric letters of the alphabet? What are the least?*

Question 13. *Can you think of any words (any language you like) that admit symmetries?*

Question 14. Give an example of a figure with exactly three symmetries. Can you generalize your example to m symmetries?

Polyhedra

A more difficult question is what are all (regular) polyhedra and what are their symmetry groups? A *polyhedron* is, roughly, a collection of polygons F_1, \dots, F_n (the ‘faces’) glued along some collection of edges. Analogous to our discussion of polygons, we will insist that our polyhedra bound a convex region in 3-space, that their faces only meet along edges, and that at exactly two faces share a common edge.



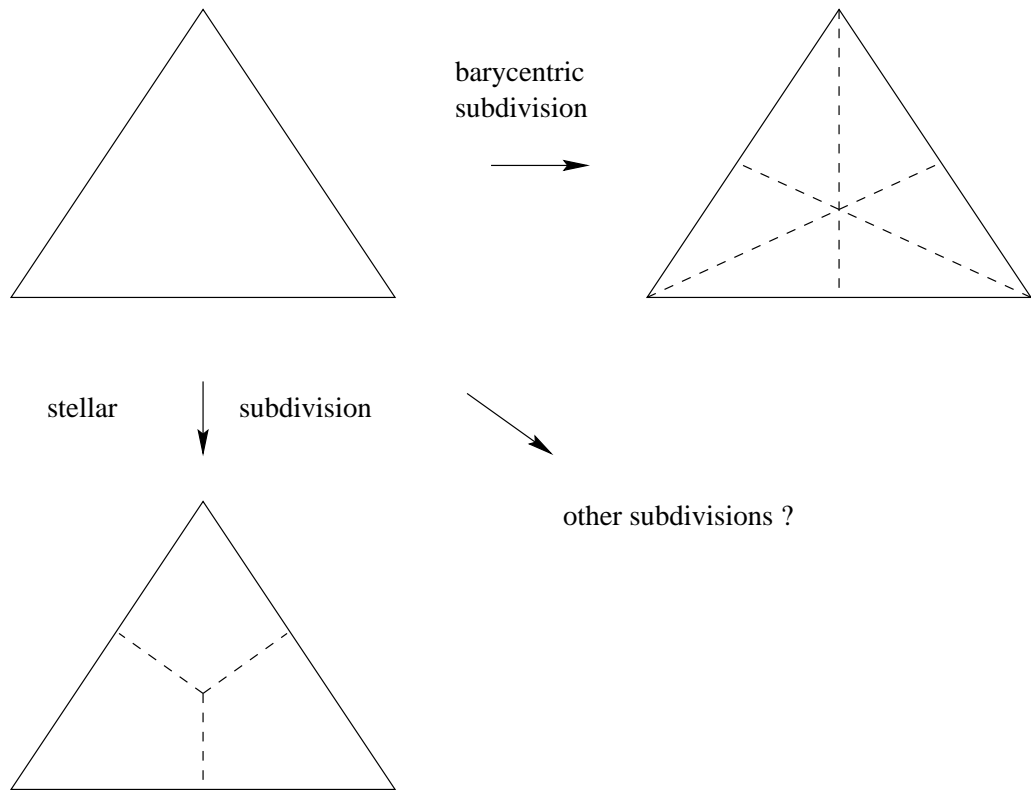
A convex polyhedron is *regular* if all of its faces are regular polygons, all faces are congruent, and every vertex is surrounded by the same number of faces.

Question 15. *What are all the regular polyhedra?*

We will discuss the problem in detail, but you should see if you can convince yourself of the answer first.

Supplemental Questions

- (1) How many planes of symmetry does each regular polyhedron have?
 - (a) cube
 - (b) tetrahedron
 - (c) octohedron
 - (d) dodecahedron
 - (e) icosahedron
- (2) How many planes of symmetry does each regular polyhedra have?
- (3) How many axes of rotational symmetry does each regular polyhedron have?
- (4) How large are the symmetry groups of these polyhedra?
- (5) Can you answer these questions about symmetry for the soccerball, the basketball, the baseball?
- (6) A *coloring* is an assignment of colors (e.g. red, blue, green, ...) to each vertex in a graph in such a way that no two vertices of the same color are adjacent (i.e. connected by an edge). How many distinct (up to symmetry) 2-colorings (i.e. only use two colors) does the 'skeleton' of the cube admit? What about 3-colorings up to symmetry?
- (7) Compute the Euler characteristic (i.e. $\#$ vertices - $\#$ edges + $\#$ faces) for each of the regular polyhedra.
- (8) Suppose each of the edges of the regular polyhedra is assigned a length of one. What are their surface areas? What are their volumes?
- (9) What might be an example of a 'quasi-regular' polyhedron?
- (10) Do similar symmetry computations for any quasi-regular polyhedra you discover.
- (11) What tessellations (i.e. tiling by polygons) of the plane can you construct? What if you have to use regular polygons? What if you can use different types of regular polygons?
- (12) What polyhedra do we obtain by performing barycentric subdivision on each of the regular polyhedra? What about stellar subdivision? What if you perform these subdivisions on tessellations? (see the illustration on the next page)



- (13) Which of these polyhedra appear in crystal configurations? What about configurations of organic molecules?
- (14) Are there any regular 'polyhedra' (the term is 'polytope' for dimensions ≥ 4) in dimension four?
- (15) How many reflections does it take to move a point in the plane to any other point in the plane? What if you are moving an equilateral triangle? What if you are moving an isosceles triangle? Any triangle?
- (16) Repeat the above exercise for tetrahedra in three dimensional space.
- (17) What happens if we only allow rotations in the previous exercises?