Fibonacci's Worksheet

Fibonacci was fond of unit fractions, i.e., fractions of the form $\frac{1}{n}$. Using the fact that juxtaposition of unit fractions and integers implies addition, translate the following Fibonacci notation to our current notation:

Example: $\frac{1}{3}\frac{1}{2}11 = 11\frac{5}{6}$

1. Show that

$$\frac{98}{100} = \frac{1}{100}\frac{1}{50}\frac{1}{5}\frac{1}{4}\frac{1}{2}$$

2. What is $\frac{1}{25}\frac{1}{5}\frac{1}{4}\frac{1}{2}$?

3. Write $\frac{77}{150}$ in terms of unit fractions, as above.

3. Write $\frac{53}{80}$ in terms of unit fractions, as above.

4. Can every fraction be written in terms of unit fractions? What is the method?

5. Fibonacci writes: *"I thought about the origin of all square numbers and discovered that they arose from the regular ascent of odd numbers. For unity [i.e., 1] is a square and from it is produced the first square, namely 1; adding 3 to this makes the second square, namely 4, whose root is 2; if to this sum is added a third odd number, namely 5, the third square will be produced, namely 9, whose root is 3; and so the sequence and series of square numbers always rise through the regular addition of odd numbers."*

(a) Demonstrate Fibonacci's idea by the following process: draw a dot. Next, add 3 more dots so that we have the outline of a square with side length 2. Next, add 5 dots to obtain another square, etc.

(b) What is the formula for the n^{th} odd term, where $a_1 = 1, a_2 = 3, a_3 = 5, \ldots$?

(c) The formula for the sum of the first n terms of an arithmetic sequence $a_1, a_2, a_3, \ldots$ (where each term is obtained from the previous term by adding the same number) is $S_n = \frac{n}{2}(a_1 + a_n)$. Find a formula for the sum of the first n odd numbers.

(d) Add the results of the previous two problems. Is this expression a perfect square?

This result is also due to Fibonacci. He writes: *"Thus, when I wish to find two square numbers whose addition produces a square number [i.e., Pythagorean triples], I take any odd square number as one of the two square numbers and I find the other square number by the addition of all the odd numbers from unity up to but excluding the odd square number. for example, I take 9 as one of the two squares mentioned; the remaining square will be obtained by the addition of all the odd numbers below 9, namely 1, 3, 5, 7, whose sum is 16, a square number, which when added to 9 gives 25, a square number."*

Worksheet #1

1. Sum up an odd number of products of successive Fibonacci numbers and compare it to the last term used [e.g., compare $(1 \cdot 1 + 1 \cdot 2 + 2 \cdot 3)$ to 3]. What is the relationship between these two values?

2. Sum up any ten consecutive Fibonacci numbers. Is this sum divisible by the seventh number in the list of the ten numbers used? What other number divides this sum (and every other such sum)?

3. Write down the eleventh term of the Fibonacci sequence. Then find a decimal approximation for $\frac{1}{89}$. How does this compare to the sum of the following decimals?

0.01
0.001
0.0002
0.00003
0.000005
0.0000008
0.00000013
0.000000021

4. For a number n of your choice, sum up the first n terms of the Fibonacci sequence. Secondly, find the $(n+2)^{\text{th}}$ term. How do these two values compare? What is the relationship between the sum of the first n terms and the $(n+2)^{\text{th}}$ term?

5. Take any four consecutive Fibonacci numbers. Find (a) the product of the first and the fourth of the numbers you chose, (b) the product of the second and third numbers, times 2, and (c) the sum of the squares of the second and third numbers. What is $a^2 + b^2$?

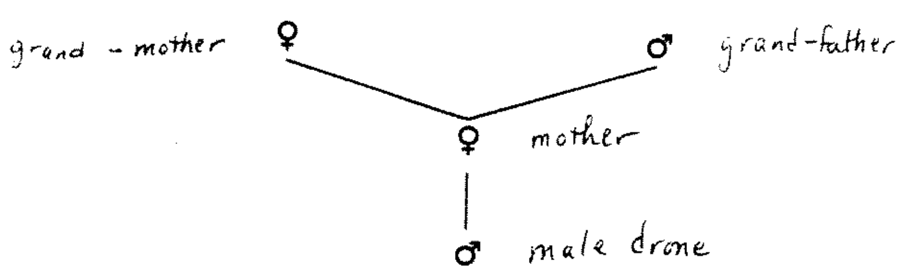
6. Take two terms of the Fibonacci sequence, f_n and f_m, where the subscript n divides the subscript m (e.g., f_3 and f_9, where $3|9$). Does the term f_n divide the term f_m (e.g., $f_3 = 2$ and $f_9 = 34$, where $2|24$) ?

7. Use the formula below to see that it gives the n^{th} term in the Fibonacci sequence.

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^n\right]$$

Worksheet #2

Let us examine the family tree of a drone, or male bee. Eggs of worker bees that are not fertilized develop into drones. Hence, a drone has no "father" and only a "mother". The queen's eggs, on the other had, are fertilized by drones and develop into females. A female bee has therefore botha "mother" and a "father". Expand the family tree of a drone, started below, seven generations back. Then answer the questions that follow.



(a) How many mothers does a drone have?

(b) How many grandparents does a drone have?

(c) How many great-grandparents does a drone have?

(d) How many great-great-grandparents does a drone have?

(f) How many great^3-grandparents does a drone have?

(g) How many great^4-grandparents does a drone have?

(h) How many great^5-grandparents does a drone have?

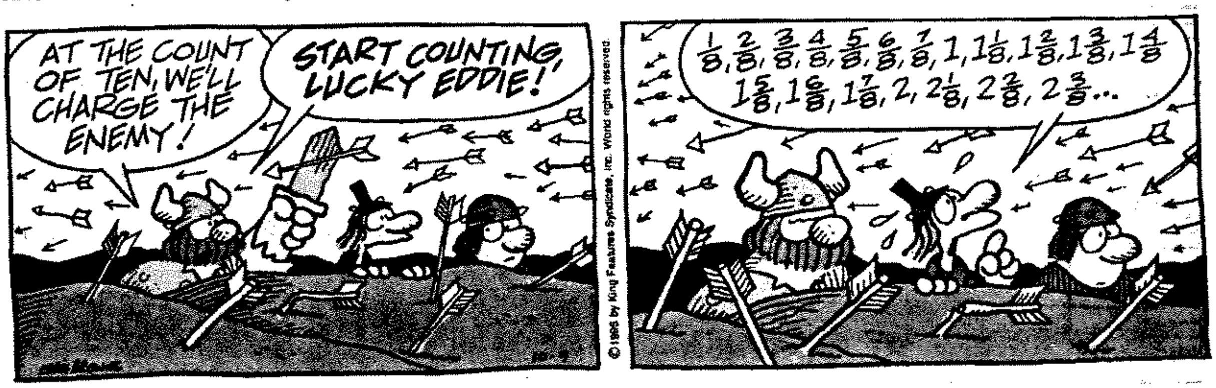
CONCLUSION:

Next consider the following problem: A child is trying to climb a staircase. The maximum number of steps he can climb at a time is two. That is, he can either climb one step or two steps at a time. If there are n steps in total, in how many different ways, C_n can he climb the staircase? The first few cases are considered below. Continue the analysis of this problem.

If $n = 1$, clearly there is only one way to climb it, so $C_1 = 1$.

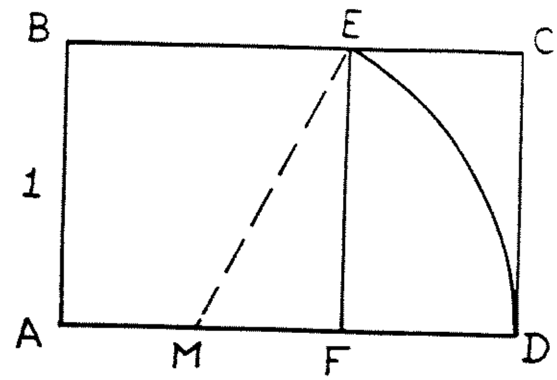
If $n = 2$, then he can either climb the two steps at once, or take them one step at a time; thus, there are two ways; i.e., $C_2 = 2$.

HAGAR THE HORRIBLE / By Chris Browne



Worksheet #2

1. Verify that ABCD is a golden rectangle



FRANK AND ERNEST

2. Find the following:

a. $\frac{1}{\phi}$

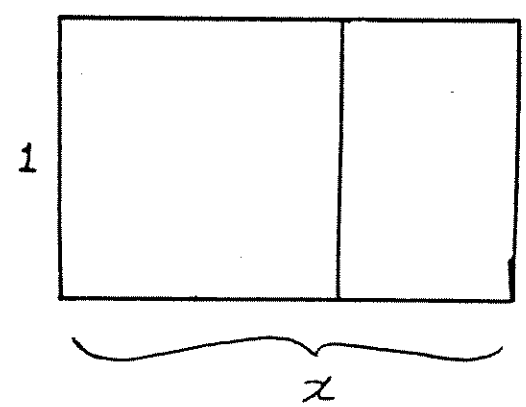
b. $\phi \cdot \frac{1}{\phi}$

c. $\phi - \frac{1}{\phi}$

d. the roots of the equation $x^2 - x - 1 = 0$.

NOTE: The exercises in a-c above can be demonstrated with a calculator as well. Put 1.6180339887 in your calculator and hit the $[x^2]$ button. What do you notice? Next, put 1.6180339887 back into your calculator and hit the $[1/x]$ button. What do you notice?

3. Prove that the golden rectangle is the only rectangle with property that cutting a square from it produces a similar rectangle.



Worksheet #3

Constructing a Golden Triangle.

Step 0 Begin with an angle of 36 degrees.

Step 1 Construct an isosceles triangle ABC whose vertex angle A is 36 degrees.

Step 2 Bisect angle B and label the intersection of the bisector with the side AC by the point D.

Step 3 Let AD = x and AB = 1. Find the lengths of BC, BD, and AD.

Step 4 There are 3 triangles: ABC, DAB, and BCD. Which 2 of these are similar triangles? (What does it mean for two triangles to be similar?)

Step 5 From the similar triangles, we have the proportion $\frac{\text{AB}}{\text{BC}} = ?$

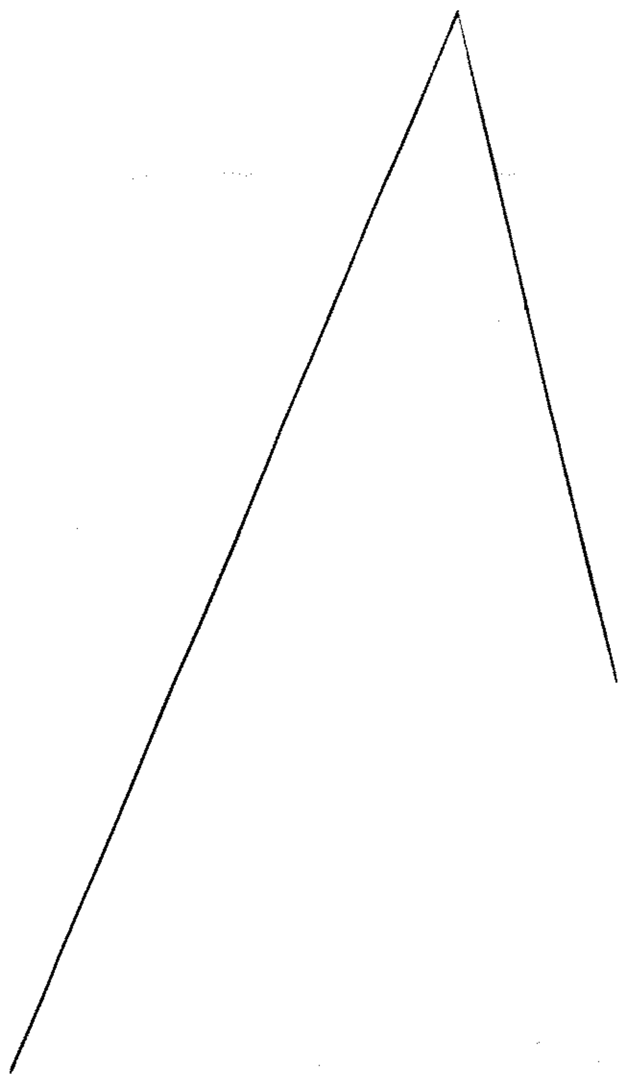
Step 6 Rewriting the proportion from Step 4, we have $\frac{1}{x} =$

Step 7 Cross multiply the proportion to obtain an equation. (Write the equation in the form $ax^2 + bx + c = 0$)

Step 8 Use the quadratic formula to solve for x. (Remember, x is the *length* AD of the side of a triangle.)

Step 9 In triangle ABC, what is the ratio of $\frac{\text{side}}{\text{base}} = \frac{1}{x}$?

Conclusion: Triangle ABC is a golden triangle.



Worksheet #4

Regular Pentagons and Pentagrams

A regular polygon is a polygon in which all of the sides have the same length and all of the interior angles have the same measure.

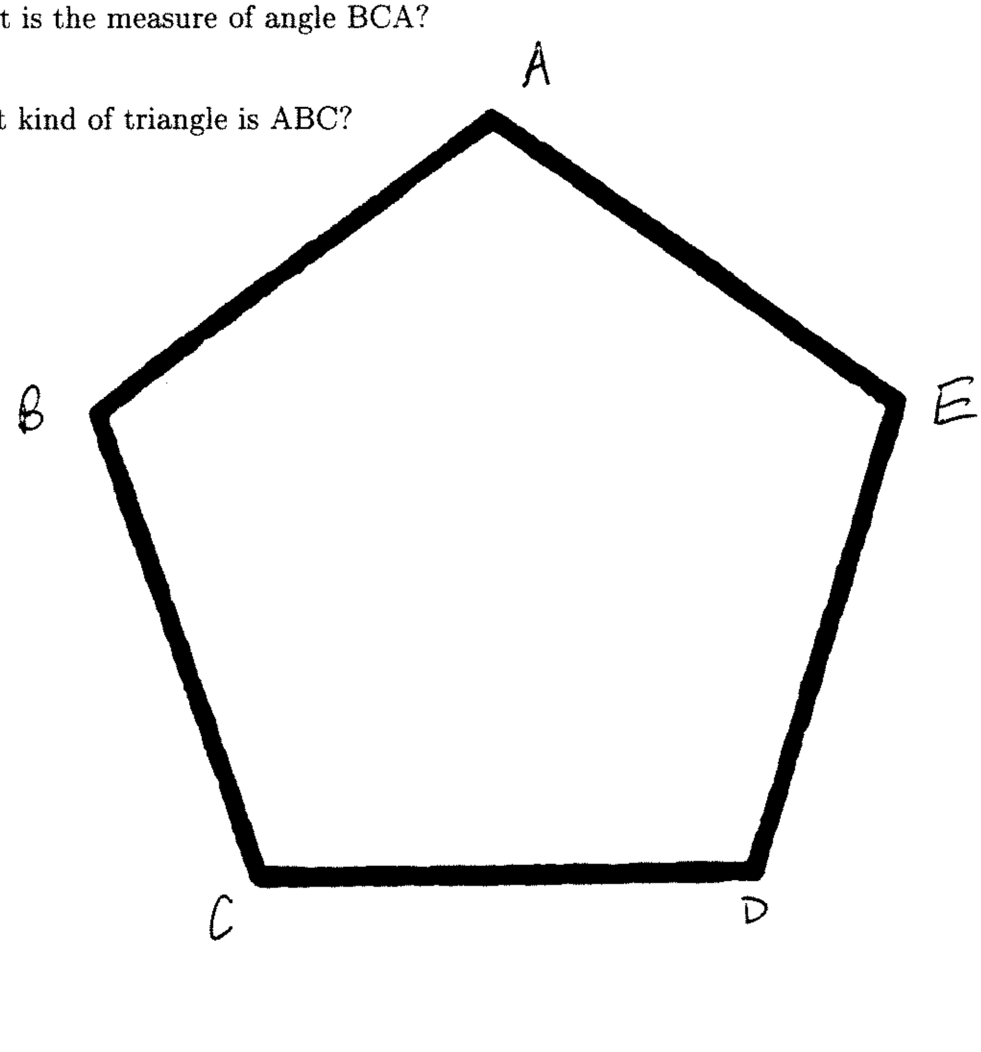
1. A regular pentagon ABCDE is shown below. Find the measure of an interior angle.

2. Draw a diagonal from vertex A to C.

a. What is the measure of angle BAC?

b. What is the measure of angle BCA?

c. What kind of triangle is ABC?

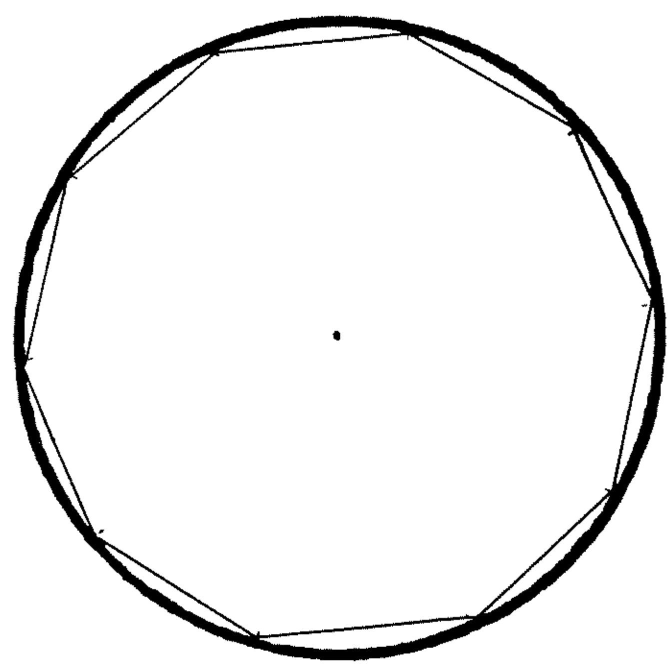


2. Draw in the remaining diagonals of the pentagon: i.e., from A to D, from B to D and E, etc. The result will be a star inscribed in the pentagram. Also, a smaller pentagon will result from the intersection points of the diagonals. Starting from the bottom-most point and going clockwise, label the points of the smaller pentagon by R, S, T, U, V.

3. The star inscribed in the original pentagon is a regular pentagram. Show that the following ratios are golden:

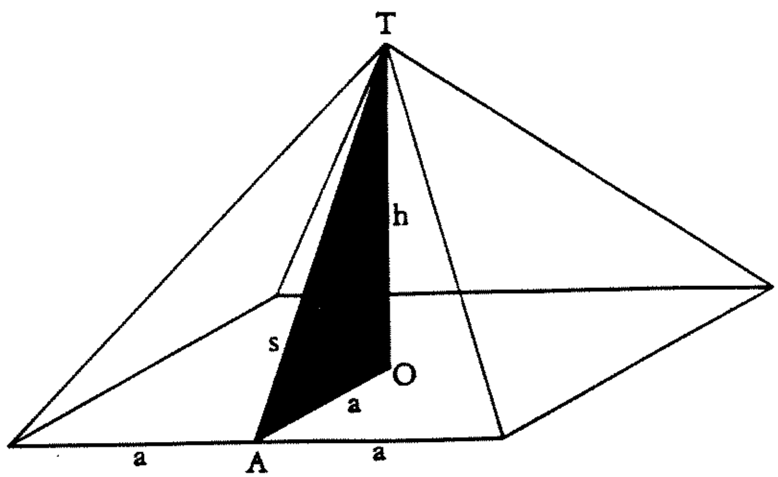
a. $\frac{AD}{AV}$ **b.** $\frac{AV}{AU}$ **c.** $\frac{AU}{UV}$

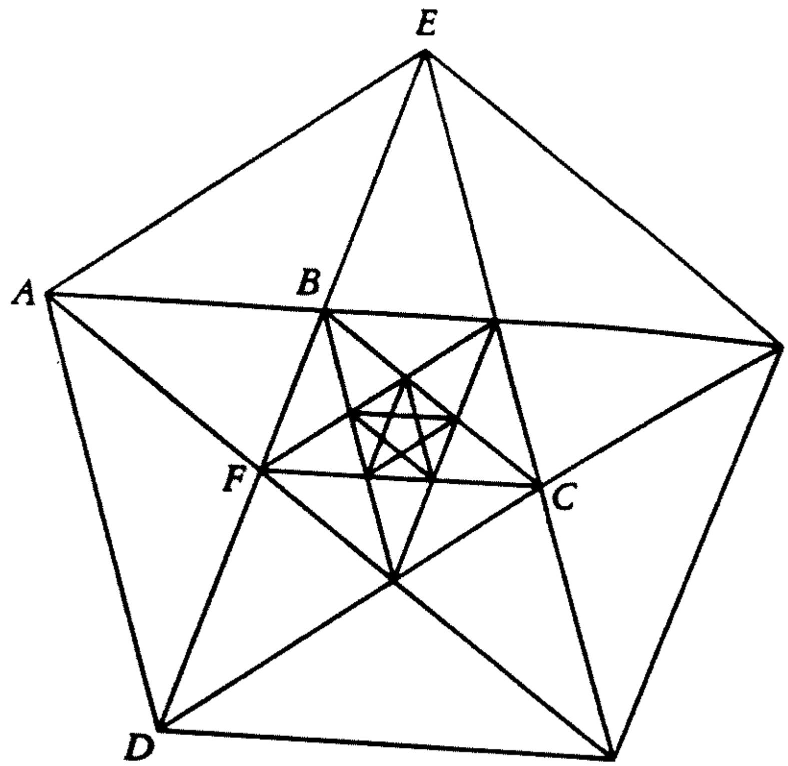
4. The decagon is also closely related to the golden ratio. The radius of the circle that circumscribes a decagon with a side length of 1 unit is equal to ϕ.



An excerpt from the book <u>Gnomon: From Pharaohs to Fractals</u> by Midhat J. Gazalé:

"It was reported that the Greek historian Herodotus learned from the Egyptian priests that the square of the Great Pyramid's height is equal to the area of its triangular lateral side."




E
A
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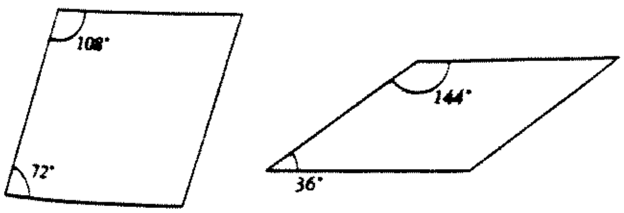
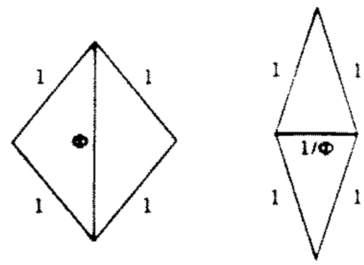
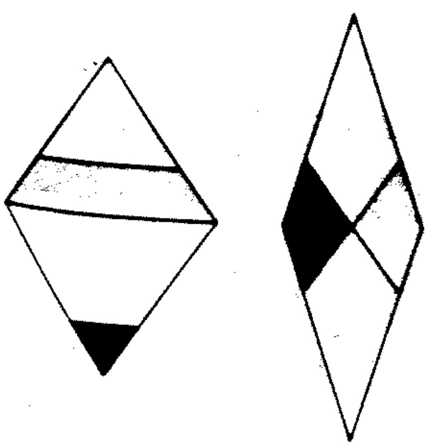


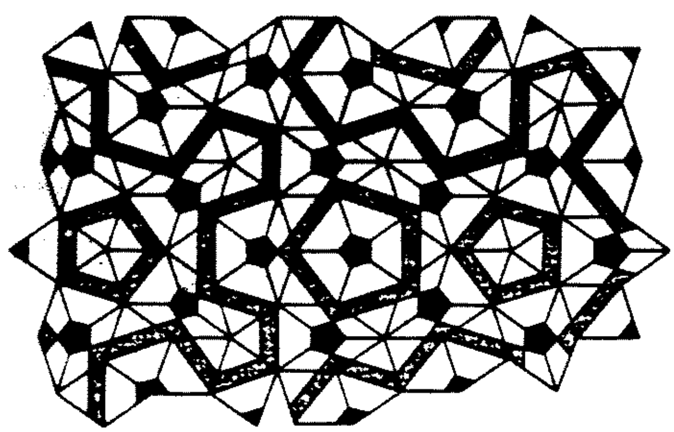
(a) (b)

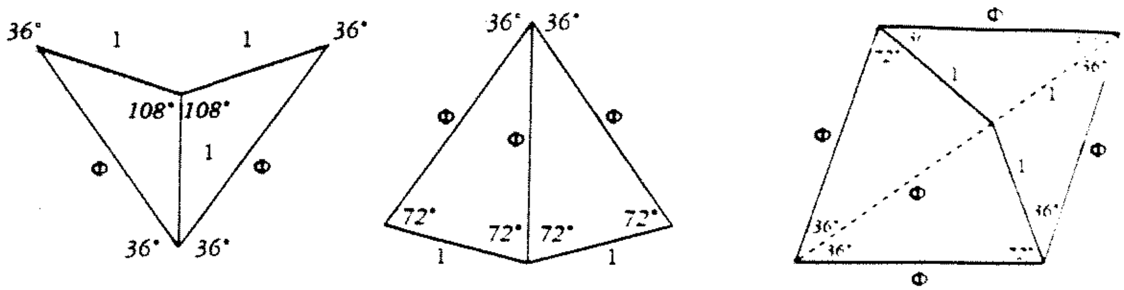


(a) (b)



(a) (b)




36°
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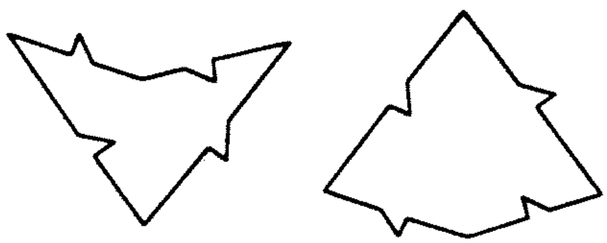
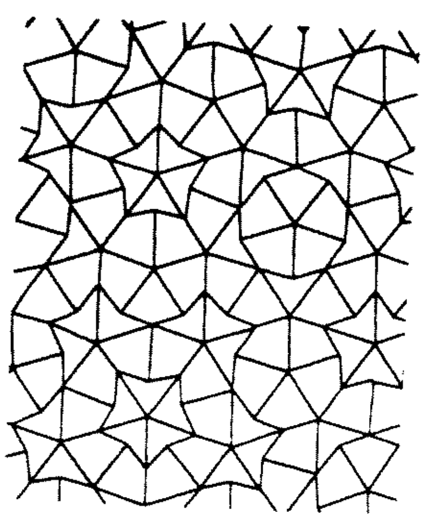
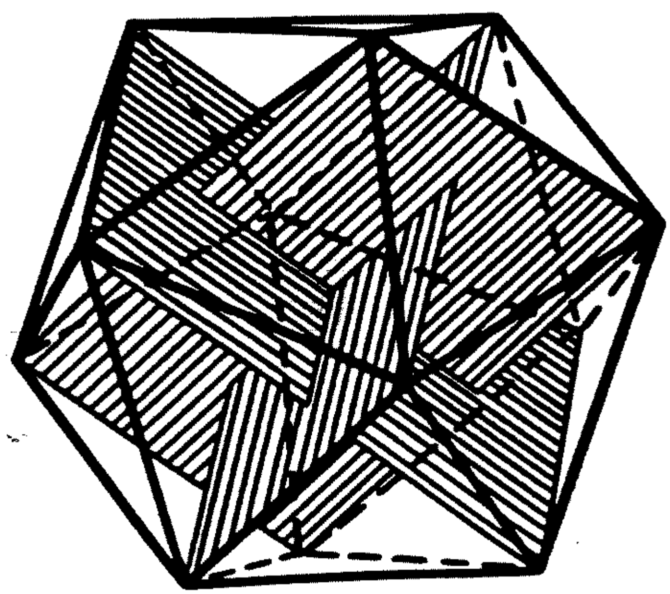
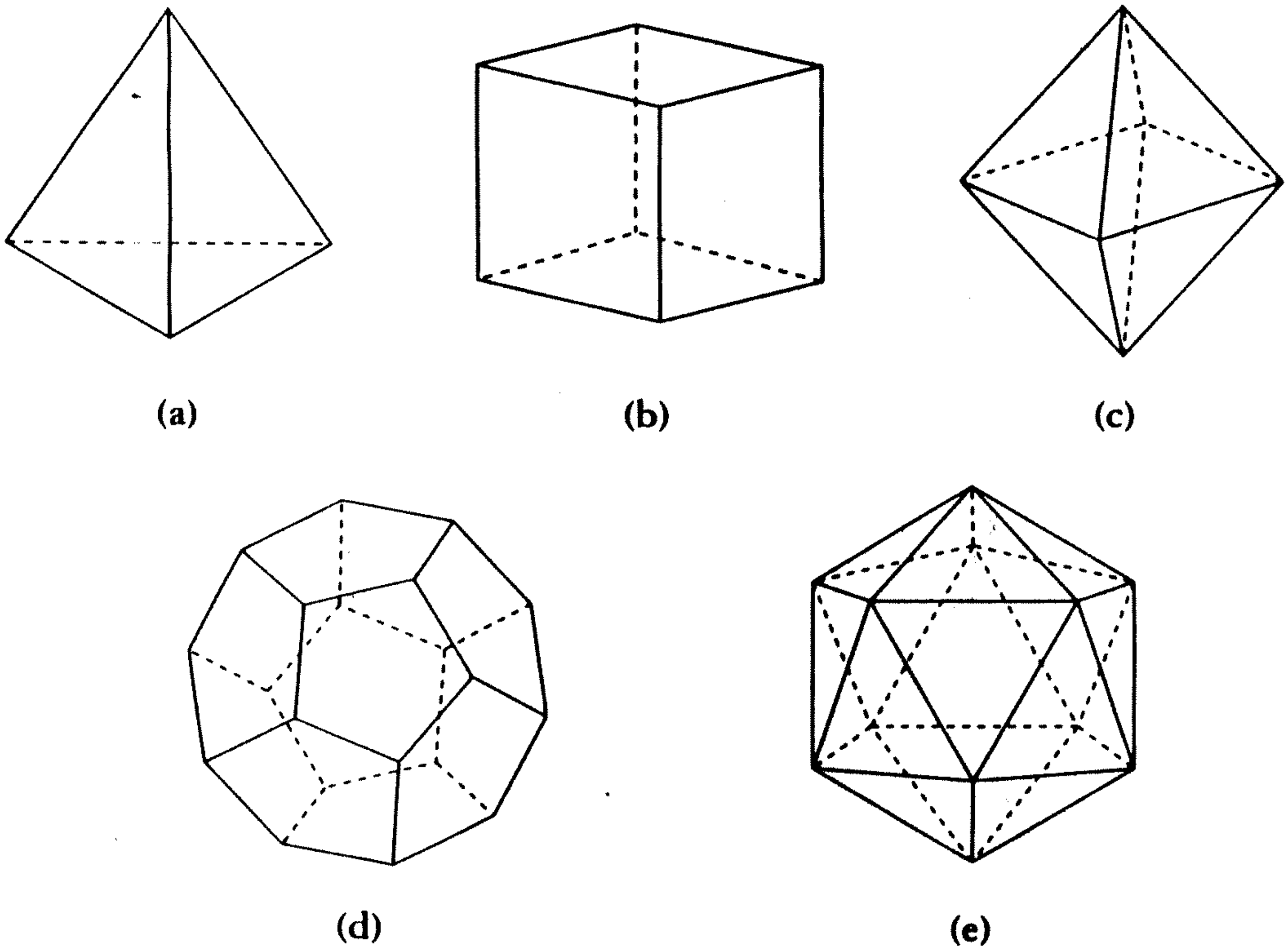
(a) (b)



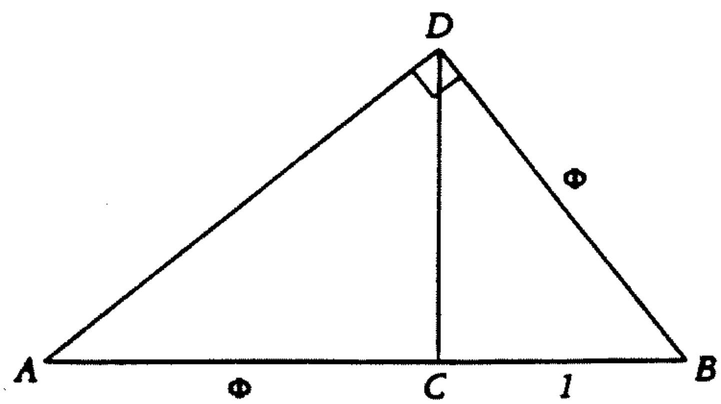
(a) (b)







(a) (b) (c) (d) (e)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