Math Circle for December 4, 2002

Rubik's Cube Part Two by David Hartenstine

Solving the Top Two Layers of the Cube

The following sequence of steps will solve the top two levels of the cube. Solving the bottom level, without messing up the part already solved, is the hardest part and involves some complicated moves. If you want to learn how to solve the cube on your own, don't read this section. Most of these steps are not too difficult to figure out with a little bit of experimenting with the cube. Also, these are the steps that I use to solve the top two rows; other methods are possible and may be more efficient.

Step 1. Choose one side (say red) to be the top and turn the cube so that the face with red in the center is on the top.

Step 2. Put each of the four edge pieces that has a red side into its correct place. When this is done the other side of each of these edge pieces should be the same color as the center piece directly below it, and there should be a red cross pattern on the top face. To move a piece into its proper position, twist the cube so that that piece is on the bottom. If the red side of that piece is facing down, twist the bottom face until the piece is on the correct face (if the other side of our piece is yellow, twist until it is underneath the yellow center piece). Now twist that face twice to put our piece in the right place. If the red side is not facing down after being moved to the bottom level, turn the cube so that red is still on top, but our red edge piece is in the front. The sequence of moves DRF^{-1} will put it on the bottom, but change its orientation so that the red side is now facing down. Now use the previous instructions. The sequence of moves reversing the orientation may move other red edge pieces out of their right places; put them back in place before proceeding.

Step 3. Now put the red corners in place one-by-one without moving any of the red edges out of place. When this is done, the top layer will be solved, and the top row of each of the outer sides should match the center piece underneath it. To do this, pick a corner on the top side and find the piece that belongs there. Place this piece in the bottom row and so that the red side faces out (and not down) by performing a sequence of moves; place all edge pieces that were moved back in their correct positions. Twist the bottom so that the red corner piece is directly below its correct location. Turn the cube so that the red side of our corner piece is facing front. If this piece is on the left, do $DL^{-1}D^{-1}L$ to put it in place. If it is on the right, do $D^{-1}R^{-1}DR$.

Step 4. The next step is to solve the middle layer. This consists of moving four edge pieces into place. The following sequence will get the first three of them right. Find a piece that belongs in the middle layer. Look on the bottom row first. If this piece is not on the bottom, find a sequence of moves that puts it on the bottom; if this messes up the top, fix it before going on. Twist the bottom so that the piece we want to put in place is not on the face where it belongs. Now twist the face where our edge piece should be so that this location is on the bottom layer. Twist the bottom so that our piece is now in position. If its orientation is wrong, start over working with the other face (if our piece belongs at $\{RF\}$ and we twisted F to move $\{RF\}$ to the bottom layer, try twisting R instead). Now twist the side back so that our piece is in the right place. This procedure moves one of the top corners out of place. To put it back and keep the edge pieces on the middle layer already put in place, twist the top and bottom so that the correct edges are out of the way when the red corner is moved back.

Step 5. To put the last edge of the middle layer in place, proceed as in Step 4 to put it in its correct place. The whole middle layer should be solved now. To get the displaced corner back without messing up the rest of the cube, twist the bottom so that the red corner is underneath where it should be. When looking straight at the red side of this displaced corner it should be on the right side. If it is not, undo the moves that put the last edge of the middle layer in place, and put it back into place by turning the face in question in the opposite direction from what you did the first time. Now do D^{-1}. Now turn the cube so that the red side of the displaced corner is facing left (and it should be in the lower left corner of the bottom row), and then do $R^{-1}FLF^{-1}RFL^{-1}F^{-1}$.

Conjugates, Commutators, and the Cube

If X and Y are any permutations, the product, XYX^{-1} is called a **conjugate** of Y (by X). In the context of Rubik's cube, X and Y would be sequences of moves. An example shows how a product of this form might be useful when solving the cube. Suppose that I know a sequence of moves that swaps the positions of two adjacent corners on the same face (and maybe moves some other pieces as well). Call this sequence of moves Y. Now I notice on my cube that there are two corners, call them 1 and 2, that I would like to interchange, but they are not on the same face, so simply doing Y will not work. However, let X be a sequence of moves that moves piece 2 to a corner adjacent to corner 1, without moving piece 1. Now if I perform X, this puts pieces 1 and 2 next to each other. The sequence Y will switch them. Then X^{-1} will move piece 1 to the place originally occupied by 2, without moving corner 2. In other words, the conjugate XYX^{-1} has switched the two corners. In this way, conjugates are useful in extending sequences of moves that require pieces to be in certain positions to configurations of the cube not in the right arrangement.

In the previous Circle, we looked at the effect of a basic move on the pieces of the cube. We saw that a basic move, say F, cycles the four corners of that side. The same is true of the edges. Now let's see what happens if we conjugate a basic move by another basic move, say U. In other words, what is the effect of the conjugate UFU^{-1} on the pieces of the cube ?

Consider the effect on the corners first. Let's label them as follows: Call $\{FUL\}$ corner 1, $\{FUR\}$ 2, $\{FRD\}$ 3, and $\{FLD\}$ 4. For the other two corners affected by UFU^{-1}, let's call $\{ULB\}$ 5 and let corner 6 refer to $\{URB\}$. Then $F = (1\,2\,3\,4)$, $U = (1\,5\,6\,2)$, and $U^{-1} = (1\,2\,6\,5)$. So the conjugate is

$$UFU^{-1} = (1\,5\,6\,2)(1\,2\,3\,4)(1\,2\,6\,5) = (2\,6\,3\,4).$$

Now let's see what happens to the edges. Label the edges on the U and F faces as follows: $\{UF\} = 1$, $\{RF\} = 2$, $\{DF\} = 3$, $\{LF\} = 4$, $\{UL\} = 5$, $\{UB\} = 6$ and $\{UR\} = 7$. Then we have that $F = (1\,2\,3\,4)$, $U = (1\,5\,6\,7)$, and $U^{-1} = (1\,7\,6\,5)$. Now compute the conjugate:

$$UFU^{-1} = (1\,5\,6\,7)(1\,2\,3\,4)(1\,7\,6\,5) = (2\,3\,4\,7).$$

Notice that for both the edges and the corners, the conjugate of F by U both of which are 4-cycles, was also a 4-cycle. F cycles the four corners on the front and U cycles the four on the top. When these basic moves are combined in this way (conjugation), four corners are again cycled. The same is true of the edges. We will see how this is useful for solving the cube shortly.

We mention the following facts about conjugates which you may want to explore on your own. If Y is an n-cycle, then any conjugate of it (i.e. any product of the form XYX^{-1} where X is any permutation) is also an n-cycle.

More generally, if Y is the product of an n-cycle and an m-cycle, so is any conjugate. Also, if you are given any two permutations Y and Z which have the same form (meaning that they have the same number of cycles and the cycles have the same length), then Y and Z are conjugate; in other words, there is a permutation X such that $Y = XZX^{-1}$.

When solving the cube, several useful sequences of moves have the form $XYX^{-1}Y^{-1}$, where X and Y are sequences of moves. This product, $XYX^{-1}Y^{-1}$, is called the **commutator** of X and Y. The commutator plays an important role in group theory. Notice that the commutator of X and Y is the conjugate of Y by X, with an additional factor Y^{-1}.

A simple example of a commutator in action when solving the cube is the sequence of moves used to put the corner pieces of the top side in their correct places. Say you want to move the corner $\{LDF\}$ to $\{LUF\}$, and you want the color facing front at $\{LDF\}$ to be facing up when it is moved to $\{LUF\}$. The sequence of moves $DL^{-1}D^{-1}L$ will accomplish this task. This is the commutator of the moves D and L^{-1}.

A much less trivial example is a sequence of moves that I use as the last step when solving the cube. Often I end up with a cube that would be solved except that the orientation of two edge pieces say on the top side is wrong. If I could flip these two pieces without moving any other piece, I would be done. Let's say I know a sequence of moves, call it X, that flips one of them, does not change anything else on the top, but messes up the rest of the cube. After carrying out X, I let Y be the move that puts the other piece I want to flip into the same place where the one I just flipped was. Then X^{-1} will undo everything that X did to the rest of the cube, flips the second one I wanted flipped, and keeps everything else on the top the way it was. By performing Y^{-1}, I move the pieces on the top back to where they were originally and the cube is solved by applying the commutator $XYX^{-1}Y^{-1}$.

Commutators are also useful when devising sequences of moves that cycle three objects, such as three corners. When devising them the following theorem is helpful.

Theorem Let X and Y be two permutations of n objects, such that there is exactly one element moved by both X and Y. In other words, there is a unique $j \in \{1, 2, \ldots, n\}$, such that $X(j) \neq j$ and $Y(j) \neq j$. Then $XYX^{-1}Y^{-1}$ moves exactly three elements.

Proof By relabeling $\{1, 2, \ldots, n\}$, we may assume that the only element moved by both X and Y is 1. We can assume (for the same reason) that

$X(1) = 2$. Because $X(1) = 2$, $X(2)$ cannot be 2, so X moves 2. Therefore, Y does not move 2. As a result, $Y(1) \neq 2$ (otherwise Y would have to move 2), so we assume that $Y(1) = 3$. Note that X does not move 3.

Now we calculate $XYX^{-1}Y^{-1}$ applied to 1. We read left-to-right. $X(1) = 2$. $Y(2) = 2$. $X^{-1}(2) = 1$, and $Y^{-1}(1) \neq 1$ (since $Y(1) \neq 1$). Therefore 1 is moved by the commutator $XYX^{-1}Y^{-1}$.

Now let j be such that $X(j) = 1$ ($j = X^{-1}(1)$. Then $j \neq 1$, and $Y(j) = j$. Now compute the commutator applied to j. $X(j) = 1$, $Y(1) = 3$, $X^{-1}(3) = 3$, and $Y^{-1}(3) = 1$. $XYX^{-1}Y^{-1}$ moves j.

Now suppose k is such that $Y(k) = 1$ ($k = Y^{-1}(1)$). Then $k \neq 1$, $j \neq k$ and $X(k) = k$. Then: $X(k) = k$, $Y(k) = 1$, $X^{-1}(1) = j$, and $Y^{-1}(j) = j$. So the commutator also moves k.

If m is not moved by either X or Y, then $XYX^{-1}Y^{-1}(m) = m$, so m is not moved by the commutator. Finally, suppose l is moved by either X or Y, say by X, and $X(l) \neq 1$ (same argument if l is moved by Y). Then we make the following observation: If l is moved by X, X must also move $X(l)$. This is because if $X(X(l)) = X(l)$, by applying X^{-1} to both sides of the equation, we see that we must have $X(l) = l$, so that X does not move l, but this is a contradiction. So, if X moves l, X must also move $X(l)$. Thus if $X(l) \neq 1$, Y does not move $X(l)$. Now compute $XYX^{-1}Y^{-1}$ applied to l. $X(l) = X(l)$, $Y(X(l)) = X(l)$, $X^{-1}(X(l)) = l$, and $Y^{-1}(l) = l$, so the commutator does not move l.

This exhausts all possibilities, so we have seen that $XYX^{-1}Y^{-1}$ moves precisely the three elements 1, $X^{-1}(1)$, and $Y^{-1}(1)$.

We can use this result and properties of conjugates to construct a useful move for cycling three corner pieces.

Exercise Find a sequence of moves that cycles the three corners $\{LUF\}$, $\{RBU\}$ and $\{LBU\}$ and does not move any other pieces. Don't worry about changing the orientation of these corners.

Solution We want to find two moves (or sequences) X and Y with the following properties: each of the corners that we want to cycle is moved by either X or Y, one of the three corners in question is moved by both X and Y, and there is no edge that is moved by both X and Y. Once we do this, we know that the commutator $XYX^{-1}Y^{-1}$ moves exactly three pieces, one of which is one of our corners. If we choose X and Y so that the other two pieces moved by the commutator are the other two corners, we will have found our move.

The move L (or L^{-1}) is a natural choice for one of the two moves, because it moves two of the corners that we want to cycle. Let $X = L$ (or L^{-1}). It is tempting to try another basic move for Y. There are three reasonable choices for this: R, U and B. Let's see if they work. If $Y = R$, there is no corner that is moved by both X and Y. If $Y = U$, two corners are moved by both permutations. If $Y = B$, more than just three corners are moved.

No basic move works for Y, so let's try something almost as simple, a conjugate of a basic move with another basic move. It makes sense to try combinations of the faces containing the corners we want to cycle. In other words, why don't we try combinations of R, U and B. The conjugate URU^{-1} cycles four corners. Two of them are $\{RBU\}$ and $\{LBU\}$. Try $Y = URU^{-1}$. Then both X and Y move the $\{LBU\}$ corner, and no other corner (or edge) is moved by both X and Y. It seems that the commutator $XYX^{-1}Y^{-1}$ will do the trick. Let's check. The commutator is the move $LURU^{-1}L^{-1}UR^{-1}U^{-1}$. By checking on the cube, we see that three corners are cycled, but they are $\{LUB\}$, $\{RUB\}$, and $\{LDB\}$. Let's try a different commutator involving X and Y. If we let $X = URU^{-1}$ and $Y = L^{-1}$, the commutator will cycle the desired three corners.

Challenge Problem The following sequences of moves come from Tom Davis's notes. They do not move any other pieces. The sequences that cycle pieces may however change the orientations of some of the pieces involved. Can you find an alternate for any of these sequences that accomplishes the same task, does not move any other pieces, and involves fewer steps?

Cycle Three Edges

a) ($\{LD\}$ $\{FD\}$ $\{RD\}$): $R^{-1}LBRL^{-1}D^2R^{-1}LBRL^{-1}$

b) ($\{UF\}$ $\{DB\}$ $\{UB\}$): $R^{-1}LU^2RL^{-1}B^2$

Cycle Three Corners

($\{LUF\}$ $\{RUB\}$ $\{LUB\}$): $URU^{-1}L^{-1}UR^{-1}U^{-1}L$

Change Orientation of Two Edges

$\{UB\}, \{UL\} \rightarrow \{BU\}, \{LU\}$:
$R^{-1}LB^2RL^{-1}D^{-1}R^{-1}LBRL^{-1}ULR^{-1}B^{-1}L^{-1}RDLR^{-1}B^2L^{-1}RU^{-1}$

Rotate Two Corners In Place

$\{URB\}$, $\{URF\} \rightarrow \{RBU\}$, $\{RFU\}$: $FD^2F^{-1}R^{-1}D^2RUR^{-1}D^2RFD^2F^{-1}U^{-1}$

These notes were based in part on those of Tom Davis, the little booklet that came with my cube, and conversations with Fletcher Gross and Nick Korevaar.