Math Circle for November 20, 2002

Rubik's Cube by David Hartenstine

Rubik's Cube is perhaps the world's most famous puzzle. It consists of six colored faces, each of which is divided into nine pieces. Each of the faces can be twisted, thus mixing up the colors of the cube. The object then is to restore the cube to its original configuration, where each face is a solid color. As most people who have played with the cube will tell you, this is not easy. We want to take a closer look at Rubik's Cube and some of the related mathematics. When reading these notes, it will be helpful to have a Rubik's Cube handy, (and maybe a screwdriver to take it apart if it becomes hopelessly messed up). Many fun facts about the cube can be found at www.rubiks.com.

Meet the Cube

Let's start by taking a closer look at our cube. Notice that there are three different types of pieces. First, we have the six center pieces, each with one color. These pieces can't be moved in relation to each other. If my cube is such that one face has red in the center and an adjacent face has white in the center, I can place the cube so that the red center piece is on top and the white one is in the front. Then, no matter how I twist the sides of the cube, I can always situate the cube so that the red center piece is again on top and the white one is in front. This is perhaps most clear if you take the pieces of the cube apart and look at the core made up of the six center pieces. When solving the cube, it is helpful to choose one color to be the top color, and another one to be the front (although in solving it, you may change your point of view sometimes). Once you have made these choices, then you have also fixed which sides are the left, right, back and bottom.

The cube also has corner pieces. There are eight of them and each has three different colors. Finally, there are the twelve remaining pieces, which we will call edge pieces; each of them has two different colored sides.

Now let's pick a top side and a front side. I usually pick red for the top and white for the front, but depending on how your cube is colored, this may not be possible. Let's call the six sides L, R, F, B, U and D, for Left, Right, Front, Back, Up and Down. We can rearrange them to get "BFUDLR", or "befuddler" which is what many people people consider the cube to be. Notice that B is Back and not Bottom. This notation gives us a convenient way to describe any sequence of moves on the cube. Every sequence of moves is built out of simpler moves: each face of the cube can be twisted clockwise or counter-clockwise, and can be twisted any multiple of 90°. Let's call any 90° twist of a face a basic move, since if we want to twist more than 90°, we can think of this as repeating a twist of 90° a certain number of times. We can identify the face we want to twist by the befuddler notation. If I want to twist the front, I will use F. Let's decide to distinguish clockwise and counter-clockwise turns in the following way: R will denote a 90° clockwise twist of the right side, and R^{-1} will be its opposite, a 90° counterclockwise twist. For example, then $U^{-1}R^{-1}LR$ describes the sequence of moves: twist the up side 90° counterclockwise, then twist the right 90° counterclockwise, then turn the left side 1/4 turn clockwise and last turn the right 1/4 turn. Then each sequence of moves, no matter how long or complicated, is built out of these twelve basic moves. We could say that these basic moves generate the set of all possible sequences of moves. Analyzing the effect of these basic moves is a logical place to start when trying to solve the cube, and we'll do this later.

We can also use this notation to describe the locations of pieces of the cube, for instance $\{UBR\}$ is the corner piece that is simultaneously on the up, back and right sides. We will use the brackets to denote locations; if we just wrote UBR, this might also refer to the move composed of twisting the up, back and the right sides 90° clockwise. Similarly, the edge pieces can be described: $\{DL\}$ is the edge that is on both the down and the left sides.

How Many Arrangements? How Hard Is It To Solve This Thing?

According to the package the cube came in, there are billions of possible configurations for the cube. Is this true? Let's see if we can figure this out.
Exercise 1 Count the total number of possible configurations for the cube, not worrying about whether each of these arrangements is possible by twisting the sides of a solved cube. This would be the number of ways to reassemble the cube after taking it apart.

Question 1 Do you think that all of these positions are attainable by turning the sides of a solved cube?

Exercise 2 Let's assume that every arrangement counted in Exercise 1 is attainable by turning the sides of a solved cube. Then let's say we are really smart when it comes to the cube, and we know how to solve it in the fewest number of basic moves from **any** scrambled position. What is the largest number of moves that we would ever have to apply to solve the cube? If you can't find this number exactly, can you find a good estimate for it? Hints: Is five enough? How could you tell? You will need to use your answer from Exercise 1.

Answer to Exercise 1 There are eight corners, so they can be placed in 8! possible arrangements, but each corner can have three different orientations (or rotations). Multiplying the number of arrangements by the number of possible orientations, we get

$$(3^8)(8!) = 264,539,520$$

ways to place the corner pieces. Now let's consider the edges. There are twelve of them, so these pieces can be located in 12! different arrangements. On top of that, each edge can be oriented in two ways, so to find the total number of ways to situate the edge pieces we have to multiply 2^{12} by 12! and we get that the edges can be placed in

$$(2^{12})(12!) = 1,961,990,554,600$$

different arrangements. Multiplying these two numbers we get the total number of possible positions for the cube:

$$P = (3^8)(8!)(2^{12})(12!) = 519,024,039,293,878,272,000 = 5 \times 10^{20}.$$

This is a huge number. No wonder solving the cube is so hard. But wait a minute. This is all the possibilities if we allow taking the cube apart. Maybe not all of these positions are possible by starting with a solved cube and twisting the faces.

Answer to Question 1 No. Not all of these are "legal" positions for the cube. In fact, if L is the number of positions that can be reached by twisting the sides of a solved cube, $L = P/12$. This is something we will return to in this week's challenge problem.

$$L = \frac{P}{12} = 43,252,003,274,489,856,000 = 4.3 \times 10^{19}.$$

This is still a huge number of arrangements, **much** more than the billions the box mentions.

Clearly, $L \leq P$. L must also divide P (i.e. P is an integer multiple of L). Why is this true? For each legal position, find the sequence of moves that produces that position (for example, $FRL^{-1}URDB^{-1}$). Compile all of these directions together into a (rather lengthy) row containing instructions to get to each legal position. Now suppose that this list does not contain every possible position (if it does, $L = P$). Now starting from our "illegal" position, apply each set of instructions from our original list to it. We will produce the same number of positions as there are legal positions. Each of these must be illegal, because if one of them was legal, we could find it in the list of legal positions, reverse the instructions and solve the cube from starting in an illegal position, but this is impossible. Let's write all of these illegal positions in a row underneath our first row. Then we have two rows, each consisting of L positions, and if this is all of the possible arrangements, then $P = L$. If not, there must be another illegal position not on our list; so we apply each set of instructions that we got for the legal positions to it, and we get L new positions, none of which we had listed previously, we can write these in a row underneath the other rows. Repeat this procedure until the total number of arrangements counted is P, then we will have that $P = Lr$, where r is the number of rows in our listing of all possible positions.

Answer to Exercise 2 Let M be this magic number. In answering this question, it is helpful to think backwards: If five moves were enough, that would mean that every position could be obtained from a solved cube in five or less moves. Since there are twelve choices for the basic moves, there are 12^5 possible sets of instructions consisting of precisely five moves, 12^4 sequences consisting of four moves, 12^3 with three moves, 12^2 with two, 12 with one and 1 move consisting of doing nothing. This is a total of

$$1 + 12 + 12^2 + 12^3 + 12^4 + 12^5 = 271,441$$

possible arrangements after at most five moves. This number is not even close to P. So since every configuration cannot be obtained with five or fewer moves from a solved cube, any arrangement not on this list can not be solved from with five or fewer moves. Note also that the true number of different arrangements after no more than five moves is less than $271,441$, since many of these arrangements will be counted twice. For example, doing nothing is the same as twisting one side and then twisting it right back.

What we are looking for then is a number, say N, such that $1+12+12^2+\cdots+12^{N-1}+12^N \geq P$, but also so that $1+12+12^2+\cdots+12^{N-2}+12^{N-1} < P$.

A little bit of calculation shows that $N = 20$. Therefore we know that M has to be at least 20; it is possible that M is larger than 20, because as we saw before, many of these arrangements are counted more than once. If we repeat the above argument with L rather than P we get that $N = 19$. So, M is at least 19, in fact, according to the little book that came with my cube, $M = 22$.

Groups, Permutations and A Simple Move

We have seen that every sequence of moves can be written as a succession of the twelve basic moves. Let's pick two such moves, say g and h.

$$g = RBD^{-1}F^{-1}U^{-1}F^{-1}$$
$$h = FBD^{-1}UR$$

How would we do the more complicated sequence of moves formed by first doing g and then h? Well, we would first do all of the steps in g, and then perform the five steps of h. We can write this as gh and think of this as a type of multiplication:

$$gh = RBD^{-1}F^{-1}U^{-1}F^{-1}FBD^{-1}UR.$$

Notice that in the middle of this product, we have an F^{-1} followed by F. These two basic moves, when done in succession, cancel each other out. A twist of any one face followed by a twist of that same face in the opposite direction amounts to doing nothing, so we could just as easily write gh as

$$gh = RBD^{-1}F^{-1}U^{-1}BD^{-1}UR.$$

Now suppose we are trying to solve the cube, and we think that g will get us closer to this goal. We do all of the steps of g, only to realize that this did not help, and we want to undo move g. How would we do this? We would have to reverse each step, so we can read the steps of g from right-to-left, but replace each twist with a twist in the opposite direction. If we do this, we get

$$g^{-1} = FUFDB^{-1}R^{-1}.$$

We call this sequence of moves g^{-1} or "g inverse" becuase it undoes g. Let's see what happens if we multiply g and g^{-1}:

$$gg^{-1} = RBD^{-1}F^{-1}U^{-1}F^{-1}FUFDB^{-1}R^{-1}.$$

We have $F^{-1}F$ in the middle so these cancel each other out. Then crossing these out, we then have U^{-1} followed by U, which cancels, leaving us with another pair $F^{-1}F$ in the middle. Continuing in this way, working our way out, we see that everything cancels out and we are left with nothing. We will also get the same result if we multiply in the opposite order $g^{-1}g$. This represents what is called the **identity**, the "sequence" of moves that moves nothing. The identity is often denoted by (1) or e. The identity has the property that for any sequence of moves g, multiplying by (1) does not change g: $g(1) = (1)g = g$. The identity plays the same role as the number 1 does in multiplication of real numbers.

Now we make the observation that if we have three moves g, h and k, then $[gh]k = g[hk]$. If we wrote both of these multiplications down with the befuddler notation, we would see that we get the same sequence, but this is perhaps even clearer if we think about what this means on the cube. $[gh]k$ means first do move $[gh]$ (which is done by first doing g and then h, and then do k, on the other hand, $g[hk]$ means first do g, and then do $[hk]$, which give exactly the same result. This means that the multiplication of moves is **associative**.

Any system with the these features (an associative multiplication, an identity, and an inverse for each element) is called a **group**. So the set of all sequences of moves is a group. Can you think of other examples of groups?

However, the order of multiplication does matter: gh is not necessarily the same as hg. For instance, if I do the move RU to a solved cube, I don't get the same thing when I do UR. This multiplication is not **commutative**.

Instead of starting by looking at complicated sequences of moves, let's look at one of the basic moves, say U. Another way to think about these moves is that any sequence of moves is a **permutation** (or rearrangement) of the corner and edge pieces. Let's see what the move U does to the corners of the cube. Only the top four corners of the cube are rearranged by U. Let's use numbers instead of the befuddler notation to label them, and call $\{ULF\}$ corner 1, $\{ULB\}$ corner 2, $\{URB\}$ 3 and $\{URF\}$ 4. Then the move U sends 1 to 2, 2 to 3, 3 to 4 and 4 back to 1. Pictorially, we could represent this as

$$\begin{array}{ccc} 1 & \rightarrow & 2 \\ \uparrow & & \downarrow \\ 4 & \leftarrow & 3 \end{array}$$

Another way to represent this permutation is by the **two-row notation**:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

This is read as follows. The corner in the top row is moved to the corner directly below it, so reading left to right we get that corner number 1 is moved to 2, 2 goes to 3, 3 goes to 4, and 4 goes to 1. One disadvantage of this notation is that each number is written twice; our notation would be more compact if we could avoid that.

One way of doing this is by using **cycle notation**. The same permutation written above in two-row notation can be written as:

$$(1\ 2\ 3\ 4)$$

We read this straight across, from left-to-right until we get to the end, so corner 1 goes to corner 2, corner 2 goes to corner 3, 3 goes to 4, and then when we get to the end, we cycle back around, so that 4 goes to 1. We could also write the same permutation as

$$(2\ 3\ 4\ 1).$$

The order of the numbers in the cycle is important, but the starting point isn't. Cycle notation is very convenient when we "multiply" two permutations. This done in the same way as multiplying sequences of moves: to multiply two permutations first perform the one on the left, then apply the second one to the result. Some examples will clarify this. Say I want to perform the move U^2, which means twist the up side of the cube 180°. By looking at the cube, we see that this will move corner 1 to corner 3, corner 2 to corner 4, corner 3 to 1, and finally 4 to 2. Written in cycle notation, this is (when restricted to the four up corners)

$$U^2 = (1\ 3)\,(2\ 4).$$

We should get the same thing when we multiply U and U:

$$UU = (1\ 2\ 3\ 4)\,(1\ 2\ 3\ 4).$$

The first application of U sends 1 to 2, but the second one sends 2 to 3, so the product moves 1 to 3. Corner 2 is moved to 3 by the first permutation and the second one takes 3 to 4, so 2 is moved to 4 by this product. Similarly, the product of U with itself sends 3 to 1 and 4 to 2, which is precisely the permutation (1 3) (2 4).

Now if we compute U^3 by multiplying U^2 and U (we can check our result on the cube):

$$U^3 = U^2U = (1\ 3)\,(2\ 4)\,(1\ 2\ 3\ 4) = (1\ 4\ 3\ 2).$$

Finally, if we take one more power of U, we get:

$$U^4 = U^3U = (1\ 4\ 3\ 2)\,(1\ 2\ 3\ 4) = (1)$$

This is the **identity** permutation, the one that doesn't move anything. This is clear from the cube, if I turn the top face 360°, every piece goes right back to where it was. We say that the move U has **order** 4. The order of any group element g is the smallest positive power n such that $g^n = (1)$.

If a permutation is written as a single cycle of length k, then its order is k. Why? If a permutation is written as a product of several cycles, where each number appears in no more than one of them, the order of the permutation is the least common multiple of the lengths of those cycles. Can you see why this is true?

Finally, we show that if a group has finitely many elements (such as the group of sequences of moves on Rubik's cube), then every element has finite order. Let g be an element of our group, and consider the list of powers of g:

$$g,\ g^2,\ g^3,\ g^4,\ \ldots\ g^n,\ \ldots$$

Since the group is finite, eventually one item in this list must be equal to one that has come before, say $g^n = g^k$ where $k < n$. Now if we multiply both sides of this equation by g^{-k}, where $g^{-k} = (g^{-1})^k$, we get that $g^{n-k} = g^{k-k} = g^0 = (1)$, so g has order $n - k$.

What About Combining Two Moves?

What happens if I do something more complicated than just twisting one side, like combining two moves of adjacent sides? We will need to keep track of the edges as well as the corners, for one thing. Let's see if we can adapt the above approach to the following questions (and not answer the questions by experimenting on the cube):

Exercise 3 What is the order of the sequence of moves R^2U^2?

Exercise 4 What is the order of the sequence of moves RU?

Answer for Exercise 3 Let's look at the corners first and look at the edges later. Label the corners in the following way: Start at corner $\{RBD\}$ and call this corner 1. Now count your way around the right side going clockwise, so that th$\{RFD\}$ is 2, $\{RFU\}$ is 3 and $\{RUB\}$ is 4. Now call $\{LUB\}$ 5 and finally $\{LFU\}$ is 6. Now all of the corners on the R and U sides have labels (drawing a sketch of the cube and labeling these corners may be helpful). Then, using cycle notation, $R = (1\ 2\ 3\ 4)$ and $U = (3\ 6\ 5\ 4)$. Then $R^2 = (1\ 3)\,(2\ 4)$, and $U^2 = (3\ 5)\,(6\ 4)$. So multiplying them, we get

$$R^2U^2 = (1\ 3)\,(2\ 4)\,(3\ 5)\,(6\ 4) = (1\ 5\ 3)\,(2\ 6\ 4).$$

If we raise this to the third power, we get the identity, so all of the corners will be in the right place.

Now, consider the edges. Let's label the edges on the right and up sides as follows: $\{BR\} = 1$, $\{DR\} = 2$, $\{FR\} = 3$, $\{UR\} = 4$, $\{UB\} = 5$, $\{LU\} = 6$, and $\{UF\} = 7$. Then $R = (1\ 2\ 3\ 4)$, $U = (4\ 7\ 6\ 5)$, $R^2 = (1\ 3)\,(2\ 4)$, and $U^2 = (4\ 6)\,(5\ 7)$.

$$R^2U^2 = (1\ 3)\,(2\ 4)\,(4\ 6)\,(5\ 7) = (1\ 3)\,(2\ 6\ 4)\,(5\ 7).$$

Since we have both 2- and 3-cycles in this permutation, we must take it to the sixth power to get the identity. So R^2U^2 has order 6 as a permutation of the edge pieces.

If we raise both permutations to the sixth power, we will get the identity for both corners and edges, so R^2U^2 has order 6.

Answer for Exercise 4 Let's try to repeat the above analysis. Note that since the multiplication of moves is not commutative, we can't just say that $(RU)^{12} = (R^2U^2)^6 = (1)$. Let's use the same labelling system (for corners and for edges) that we used for Exercise 3.

First consider the corners:

$$RU = (1\ 2\ 3\ 4)\,(3\ 6\ 5\ 4) = (1\ 2\ 6\ 5\ 4)$$

so the move RU has order 5 as a permutation on the corners.

Now the edges:

$$RU = (1\ 2\ 3\ 4)\,(4\ 7\ 6\ 5) = (1\ 2\ 3\ 7\ 6\ 5\ 4).$$

This is a cycle of length 7, so if we raise it to the seventh power, we get (1). So, if we perform RU seven times, all of the edge pieces should be in place. Therefore, if we do the move RU 35 times, all of the corners and edges should be back where they were when we started.

Starting from a solved cube, our work indicates that if we perform the move R^2U^2 six times , we should end up with a solved cube. Let's try it. It works, great! We were right.

Now let's (careful, it is easy to make a mistake) perform the move RU 35 times on a solved cube. We should end up with a solved cube again, right? If we do this move exactly 35 times, we don't get a solved cube. What went wrong? If we look closely, we will see that every piece is in the right place, but the corners have been rotated out of position. We did not take orientation into account when we computed the order to be 35. The 35 applications of RU have twisted each corner 120° out of place. To get them to have the correct orientation, we will have to do RU 70 more times, for a total of 105. We need to do this 70 more times, because the first 35 times will rotate the corners another 120°, so they will still be out of position, and it will take 35 more times to get them right.

An Approach for Solving the Cube

Most people solve the cube in stages. They start by solving one level, say the top side, but at the same time they put all of the pieces in the top row around the outside in the right place. When this is done, the top side should be all one color, and the top row of each side (except the bottom) should be the same color as the center piece. In other words, the top layer of the cube is completely solved. Now work down: the next step is to get the middle layer solved. This is harder than the previous step because not only do you want to solve the middle layer, you want to do so in such a way that you don't mess up the top layer. Once this is done, all that is left is to take care of the bottom layer, but this is the hardest part! You have to move all of the remaining pieces into their right places, without screwing up the part of the cube that is already solved.

Different people have different approaches to carrying out this procedure. Some people like to put the edge pieces in place first and worry about the corners last, others do it the other way. If you want complete instructions for solving the cube, take a look at the little booklet that came with the cube, or visit the website mentioned at the beginning. For some detailed hints, look at the Davis notes.

There are other ways to solve the cube as well, and algorithms that have been developed to solve the cube quickly on a computer do not follow this approach – rather than solving parts of the cube in stages (which is easier for us humans), the whole cube is worked with at the same time. If you looked at this procedure step-by-step, it would not seem to be getting close to a solution until the cube is almost completely solved.

Challenge Problem We stated that L, the number of legal positions of the cube, is the same as $P/12$, where P is the total number of possible arrangements of the cube if we are allowed to take it apart. This problem asks you to partially show why this is true, by showing that $L \leq P/3$. This can be done by considering the orientations of the corners when the cube is in a legal position.

Take the cube in any position (jumbled or not), and choose two opposite faces, like front and back. Say their colors are yellow and white. Then no corner piece has both yellow and white on it, and every corner has one of these two colors. Let's look at the white side first. At each corner, measure how far in degrees that corner would have to be rotated (counterclockwise) in order to put a white or yellow square on that side. For each corner, the possible answers are 0, 120 or 240 (= -120). Now do the same thing for the yellow side. Add all of these numbers up. Your result should be a multiple of 120. Do you notice anything special about this number? What happens if you perform a sequence of moves on the cube and repeat the above exercise? Can you make a conjecture and prove it?

Use your observations to prove that $L \leq P/3$.

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