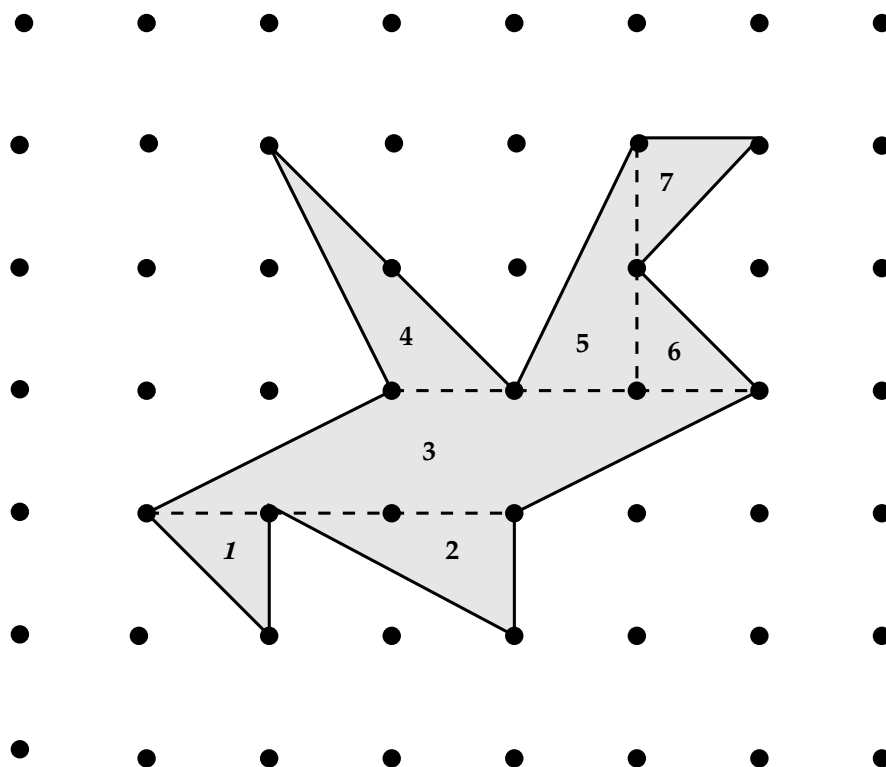# PICK'S THEOREM

PROBLEM: Find the area of a polygon whose vertices lie on a unitary square grid.

For example:

$$A = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 = \frac{1}{2} + 1 + 3 + 1 + 1 + \frac{1}{2} + \frac{1}{2} = 7\frac{1}{2}$$

There is a conceptually easy, but laborious solution: I can chop my complicated polygon into easier shapes, compute the areas of these easy shapes and add them up. In the previous example, the problem reduces to computing the area of six triangles and one parallelogram.

But now imagine a 100,000 × 100,000 square grid, and start to zig-zag like crazy to obtain a monstrously complicated shape...would you really want to go through the previously described process?

Fortunately we don't have to. Mr. Pick tells us that there is a nice, beautiful, easy formula that tells us the area of the polygon if we know:

1. the number of grid points inside the polygon (which we'll denote by I);

2. the number of grid points on the boundary of the polygon (which will be denoted by B).

In our example:

$$I = 2, B = 13$$

So from these two guys we should be able to cook up 7 and 1/2...mmmh...easier said than done, Mr. Pick!

Let's try and figure out this mysterious formula. Let's first try out some extremely easy shapes and see if we spot any pattern:
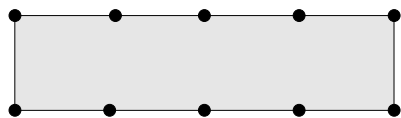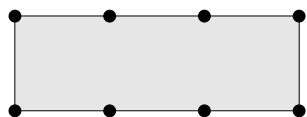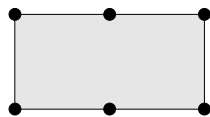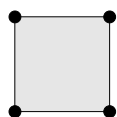
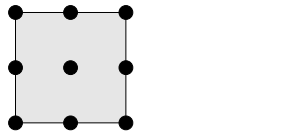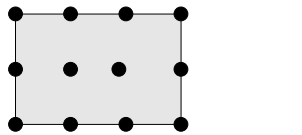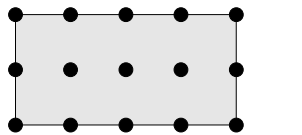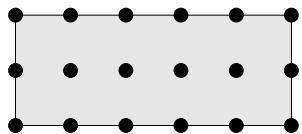| A	B	I
1	4	0
2	6	0
3	8	0
4	10	0

There seems to be some sort of pattern: every time I make my rectangle one unit larger, the area increases by one and the number of boundary points by two. In a formula, this means:

$$B = 2A + 2$$

$$A = \frac{B}{2} - 1$$

Can this be the right formula? Our hopes are soon dismissed...if we check it on our first example, it doesn't turn out 7 and 1/2. (bummer)

But there is one conceptual thing to observe. None of our objects had any interior points. Let's then again experiment on a class of extremely easy objects that do have interior points and see what happens:

| A	B	I	B/2-1
4	8	1	3
6	10	2	4
8	12	3	5
10	14	4	6

A-HAH!! Clear like the sun on a sunny day!

$$A = \frac{B}{2} - 1 + I.$$

Is this it? Indeed, if we test it on our initial monster polygon we get the desired result.

But this does not mean much. It may be just out of sheer luck that the right result comes out in our particular example. We need to sit down and find a PROOF of the formula.

The key observation is the following: suppose we start from a big polygon R and we chop it into two smaller ones by joining two vertices, as in the picture:
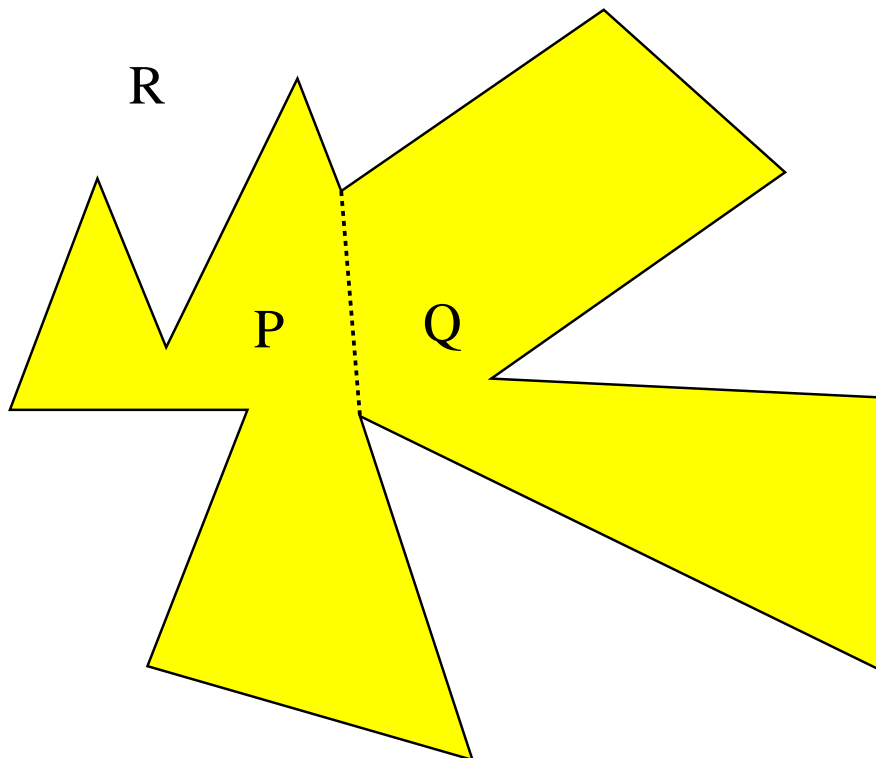
Let's denote our "wannabe" formula by "Pick(*polygon*)" (i.e. $Pick(R) = \frac{B_R}{2} - 1 + I_R$).

Then I claim that

$$Pick(R) = Pick(P) + Pick(Q).$$

This I will let you figure out on your own. It's not too hard, but it requires keeping track carefully of the number of interior points you lose and the number of new boundary points you introduce in making the subdivision.

And now it is easy easy easy! Right? Well, not quite that easy, but what is left is just a few observations that will reduce the problem to an extremely easy one:

1. Every polygon, no matter how ugly, can be chopped up into triangles. Hence if we prove that the formula is true for triangles, then by the previous observation it will be true for any polygon.

2. Every triangle can be inscribed in a rectangle with sides parallel to the grid, as in this picture:
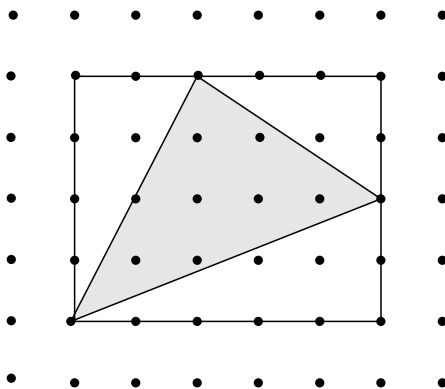
(Be careful: there are some triangles for which this construction "degenerates": can you figure out which ones and what does the decomposition look like?)

Hence, if we prove the formula for rectangles and right triangles then it will hold for all triangles.

3. Every right triangle is "half" of a rectangle. So we really need to show the formula for rectangles.

4. Every rectangle can be chopped up into unit squares. So we only need to check that the formula holds for the unit square. But we already did that (See the second picture in this handout!!!). So we are done!!!

(Isn't that cute?)

THE END

Inspiration and ideas for this handout come mostly from Ian Stewart's book "Another fine math you've got me into". Specifically, Pick's theorem is treated in the section "How many Goats in the Orchard".