

5 The Jones' polynomial

In the early 90's a mathematician at UC Berkeley named Vaughn Jones (a native of New Zealand) discovered a new way to tell knots apart. For this work he received the Fields Medal, the highest award in mathematics (equivalent to the Nobel Prize). He assigns a polynomial to every knot. If the polynomials are different, the knots are also different. It is possible for different knots to have the same Jones' polynomial, but it happens rarely. This polynomial is a remarkably good method of distinguishing knots. We will go through the construction of this polynomial that is due to Louis Kauffman.

The first step is to assign a *bracket* $\langle K \rangle$ to every knot (or link) diagram K . This is going to be a polynomial, initially in variables A, B, C , and it will satisfy the following rules.

$$\text{Rule 1: } \langle \bigcirc \rangle = 1$$

$$\text{Rule 2: } \langle \nearrow \searrow \rangle = A \langle \rangle + B \langle \frown \rangle$$

$$\langle \searrow \nearrow \rangle = A \langle \smile \rangle + B \langle \rangle$$

$$\text{Rule 3: } \langle L \bigcirc \rangle = C \langle L \rangle$$

Bracket rules

Now we want to make sure that things don't change when we perform a Reidemeister move.

17. Using Rules 1-3 express $\langle \bigcirc \rangle$ in terms of $\langle \rangle$ and $\langle \frown \rangle$. Deduce: To ensure that the bracket does not change under the Reidemeister move of type II we need the following relations between A, B, C :

$$A^2 + ABC + B^2 = 0, \quad BA = 1$$

We will now put $B = A^{-1}$ and $C = -A^2 - A^{-2}$ and our Rules 1-3 become:

Rule 1: $\langle \bigcirc \rangle = 1$

Rule 2: $\langle \diagdown \rangle = A \langle \rangle \quad \langle \rangle + A^{-1} \langle \diagup \rangle$

$\langle \diagup \rangle = A \langle \diagdown \rangle + A^{-1} \langle \rangle$

Rule 3: $\langle L \bigcirc \rangle = (-A^2 - A^{-2}) \langle L \rangle$
Bracket rules

18. Check that the bracket does not change after a Type III Reidemeister move.

19. Compute the bracket of the unlink with 2 components and with 3 components.

20. Compute the bracket of the Hopf link. You should get $-A^4 - A^{-4}$.

There is still the Type I Reidemeister move to worry about.

21. Express $\langle \wp \rangle$ in terms of $\langle \setminus \rangle$

You should get

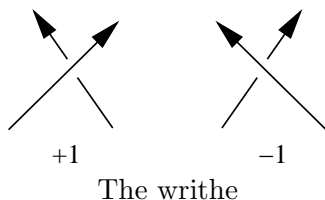
$$\langle \wp \rangle = -A^3 \langle \setminus \rangle$$

OK, so we have a problem! The bracket is not invariant under the Type I Reidemeister move. There are several possible remedies.

- We could find a number A so that $-A^3 = 1$. This would give us a numeric invariant of knots. For example $A = -1$ would work. There are more interesting choices for A ; however, you would have to know about complex numbers.
- Another numeric invariant would be the *span* of the bracket polynomial, i.e. the difference between the highest and the lowest powers of A .
- The most interesting resolution of this problem involves the notion of the *writhe* of a knot projection.

6 Writhe


This is very similar to the concept of the linking number except that we are “color-blind”. Orient all strings in the diagram and count with signs according to the right-hand rule.



22. What is the writhe of the standard picture of the trefoil? What is the writhe of the unknot?

23. How does the writhe change if we reverse all orientations?

24. How does the writhe change under the Reidemeister moves?

25. Show that no matter how hard you try you will not be able to transform  into the circle (with no crossings) without using at least one Reidemeister move of Type I.

26. Show that

$$X(L) = (-A^3)^{-w(L)} \langle L \rangle$$

is unchanged under all Reidemeister moves. Here L is a knot projection and $w(L)$ is the writhe of L .

27. Compute $X(L)$ for the trefoil.

28. Compute $X(L)$ for the figure 8 knot. Deduce that the figure 8 knot is really a knot!

29. What happens to $X(L)$ if we reverse all crossings?

Deduce that the trefoil is not the same as its mirror image. We say that trefoil is *chiral*. Recall that the figure 8 knot *is* the same as its mirror image. It is *achiral*.

The polynomial $X(L)$ is essentially the same as the original Jones' polynomial (replacing A by $t^{-\frac{1}{4}}$ would give the original version). You have seen that it is hard work to compute it, particularly if there are many crossings. But computers can handle this computation without much trouble. There is a program called *knotscape* (mathematicians' sense of humor) that performs this task.

7 Resources

- Colin C. Adams: *The knot book*, W.H. Freeman and Company, New York, 1999

Most topics we discussed and many more are in this very accessible book. Pick up a copy and have fun!

- <http://www.cs.ubc.ca/nest/imager/contributions/scharein/KnotPlot.html>

Robert Scharein's excellent web site, packed with cool pictures, movies, and further links (no pun intended!). He is the author of *knotplot*, software that produces such pictures. You can download the program for free.

- <http://www.math.utk.edu/~morwen/knotscape.html>

Morwen Thistlethwaite's program that computes various knot polynomials. You draw a knot with the mouse and it computes the polynomials.

- <http://www.math.uic.edu/~kauffman/Tots/Knots.htm>

Louis Kauffman's tutorial on knots and "bracketology". Try to read it. At least the first half should be accessible.