Mathematical Induction

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1 Knocking Down Dominoes

The natural numbers, $\mathcal{N}$, is the set of all non-negative integers:

$$\mathcal{N} = \{0, 1, 2, 3, \ldots\}.$$

Quite often we wish to prove some mathematical statement about every member of $\mathcal{N}$. As a very simple example, consider the following problem:

Show that

$$0 + 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \tag{1}$$

for every $n \geq 0$.

In a sense, the above statement represents a infinity of different statements; for every n you care to plug in, you get a different "theorem". Here are the first few:

$$\begin{aligned} 0 &= 0(1)/2 = 0 \\ 0+1 &= 1(2)/2 = 1 \\ 0+1+2 &= 2(3)/2 = 3 \\ 0+1+2+3 &= 3(4)/2 = 6 \end{aligned}$$

and so on. Any one of the particular formulas above is easy to prove—just add up the numbers on the left and calculate the product on the right and verify that they are the same. But how do you show that the statement is true for *every* $n \geq 0$? A very powerful method is known as mathematical induction, often called simply "induction".

A nice way to think about induction is as follows. Imagine that each of the statements corresponding to a different value of n is a domino standing on end. Imagine also that when a domino's statement is proven, that domino is knocked down.

We can prove the statement for every n if we can show that every domino can be knocked over. If we knock them over one at a time, we'll never finish, but imagine that we can somehow set up the dominoes in a line and close enough together that when domino number k falls over, it knocks over domino number $k + 1$ for every value of k. In other words, if domino number 0 falls, it knocks over domino 1. Similarly, 1 knocks over 2, 2 knocks over 3, and so on. If we knock down number 0, it's clear that all the dominoes will eventually fall.

So a complete proof of the statement for every value of n can be made in two steps: first, show that if the statement is true for any given value, it will be true for the next, and second, show that it is true for $n = 0$, the first value.

What follows is a complete proof of statement 1:

Suppose that the statement happens to be true for a particular value of n, say $n = k$. Then we have:

$$0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}. \tag{2}$$

We would like to start from this, and somehow convince ourselves that the statment is also true for the next value: $n = k + 1$. Well, what does statement 1 look like when $n = k + 1$? Just plug in $k + 1$ and see:

$$0 + 1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}. \tag{3}$$

Notice that the left hand side of equation 3 is the same as the left hand side of equation 2 except that there is an extra $k+1$ added to it. So *if* equation 2 is true, then we can add $k+1$ to both sides of it and get:

$$0+1+2+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}. \tag{4}$$

showing that if we apply a little bit of algebra to the right hand side of equation 4 it is clearly equal to $(k+1)(k+2)/2$ — exactly what it should be to make equation 3 true. We have effectively shown here that if domino k falls, so does domino $k+1$.

To complete the proof, we simply have to knock down the first domino, domino number 0. To do so, simply plug $n=0$ into the original equation and verify that if you add all the integers from 0 to 0, you get $0(0+1)/2$.

Sometimes you need to prove theorems about all the integers bigger than some number. For example, suppose you would like to show that some statement is true for all polygons (see problem 10 below, for example). In this case, the simplest polygon is a triangle, so if you want to use induction on the number of sides, the smallest example that you'll be able to look at is a polygon with three sides. In this case, you will prove the theorem for the case $n=3$ and also show that the case for $n=k$ implies the case for $n=k+1$. What you're effectively doing is starting by knocking down domino number 3 instead of domino number 0.

2 Official Definition of Induction

Here is a more formal definition of induction, but if you look closely at it, you'll see that it's just a restatement of the dominoes definition:

Let $\mathcal{S}(n)$ be any statement about a natural number n. If $\mathcal{S}(0)$ is true and if you can show that if $\mathcal{S}(k)$ is true then $\mathcal{S}(k+1)$ is also true, then $\mathcal{S}(n)$ is true for every $n \in \mathcal{N}$.

A stronger statement (sometimes called "strong induction") that is sometimes easier to work with is this:

Let $\mathcal{S}(n)$ be any statement about a natural number n. To show using strong induction that $\mathcal{S}(n)$ is true for all $n \geq 0$ we must do this: If we assume that $\mathcal{S}(m)$ is true for all $0 \leq m < k$ then we can show that $\mathcal{S}(k)$ is also true.

The only difference between these two formulations is that the former requires that you get from the statement about k to the statement about $k+1$; the latter lets you get from any previous step (or combination of steps) to the next one. Notice also that the second formulation seems to leave out the part about $\mathcal{S}(0)$, but it really doesn't. It requires that you be able to prove $\mathcal{S}(0)$ using no other information, since there are no natural numbers n such that $n < 0$.

Using the second formulation, let's show that any integer greater than 1 can be factored into a product of primes. (This does not show that the prime factorization is unique; it only shows that some such factorization is possible.)

To prove it, we need to show that if all numbers less than k have a prime factorization, so does k. If $k=0$ or $k=1$ we are done, since the statement of the theorem specifically states that only numbers larger than 1 are considered. If k is prime, it is already a product of prime factors, so we're done, and if $k=pq$, where p and q are non-trivial factors, we know that $p < k$ and $q < k$. By the induction hypothesis, both p and q have prime factorizations, so the product of all the primes that multiply to give p and q will give k, so k also has a prime factorization.

3 Recursion

In computer science, particularly, the idea of induction usually comes up in a form known as recursion. Recursion (sometimes known as "divide and conquer") is a method that breaks a large (hard) problem into parts that are smaller, and usually simpler to solve. If you can show that any problem can be subdivided into smaller ones, and that the smallest problems can be solved, you have a method to solve a problem of any size. Obviously, you can prove this using induction.

Here's a simple example. Suppose you are given the coordinates of the vertices of a simple polygon (a polygon whose vertices are distinct and whose sides don't cross each other), and you would like to subdivide the polygon into triangles. If you can write a program that breaks any large polygon (any polygon with 4 or more sides) into two smaller polygons, then you know you can triangulate the entire thing. Divide your original (big) polygon into two smaller ones, and then repeatedly apply the process to the smaller ones you get.

The concept of recursion is not unique to computer science—there are plenty of purely mathematical examples. Here's one of the most interesting that you may wish to play with:

Ackermann's function is defined as follows on all pairs of natural numbers:

$$\begin{aligned}
A(0,n) &= n+1 \\
A(m,0) &= A(m-1,1), && \text{if } m>0 \\
A(m,n) &= A(m-1,A(m,n-1)), && \text{if } m,n>0
\end{aligned}$$

Just for fun, try to calculate $A(4,2)$. (Hint: First figure out what $A(0,n)$ looks like for all n. Then figure out what $A(1,n)$ looks like, for all n, et cetera.)

4 Make Up Your Own Induction Problems

In most introductory algebra books there are a whole bunch of problems that look like problem 1 in the next section. They add up a bunch of similar polynomial terms on one side, and have a more complicated polynomial on the other. In problem 1, each term is k^2. Just add them up for $k = 0, 1, \ldots, n$.

Here's how to work out the term on the right. Let's do:

$$S(n) = 0\cdot 1\cdot 2 + 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + n\cdot(n+1)\cdot(n+2).$$

Work out the value of $S(n)$ by hand for a few values of $n = 0, 1, 2, \ldots$. The first few $S(n)$ values are: $0, 6, 30, 90, 210, 420, 756, 1260$.

Now list those in a row and take successive differences:

0		6		30		90		210		420		756		1260
	6		24		60		120		210		336		504	
		18		36		60		90		126		168		
			18		24		30		36		42			
				6		6		6		6				
					0		0		0					

Notice that other than the top line, every number on the table is the difference between the two numbers above it to the left and right. If all the terms in your sum are generated by a polynomial, you'll eventually get a row of all zeroes as in the example above. Obviously if we continued, we'd have row after row of zeros.

Now look at the non-zero numbers down the left edge: $0, 6, 18, 18, 6, 0, 0, \ldots$, and using those numbers, calculate:

$$S(n) = 0\binom{n}{0} + 6\binom{n}{1} + 18\binom{n}{2} + 18\binom{n}{3} + 6\binom{n}{4} + 0\binom{n}{5} + 0\binom{n}{6} + \cdots. \tag{5}$$

Remember that $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{2} = n(n-1)/2!$, $\binom{n}{3} = n(n-1)(n-2)/3!$, $\binom{n}{4} = n(n-1)(n-2)(n-3)/4!$, and so on.

Equation 5 becomes:

$$S(n) = 0 + 6n + \frac{18n(n-1)}{2} + \frac{18n(n-1)(n-2)}{6} + \frac{6n(n-1)(n-2)(n-3)}{24}.$$

A little algebra converts the equation above to the simplified form below. Check that it works for the first few values of n, and if you wish, construct a standard proof by induction that it works:

$$S(n) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

If you're really ambitious, you can even show that the technique above (summing the coefficients in the left diagonal by various factors of $\binom{n}{k}$) works, using induction.

5 Exercises

These problems are all related, and are all pretty mechanical. You may wish to do a few of them just to exercise your algebra and a mechanical application of induction. Some involve a lot of grinding—they're mechanical, not necessarily easy!

Each series below has n terms:

$$\begin{aligned}
0^1 + 1^1 + 2^1 + 3^1 + \cdots + (n-1)^1 &= \frac{n^2}{2} - \frac{n}{2} \\
0^2 + 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 &= \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6} \\
0^3 + 1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 &= \frac{n^4}{4} - \frac{n^3}{2} + \frac{n^2}{4} \\
0^4 + 1^4 + 2^4 + 3^4 + \cdots + (n-1)^4 &= \frac{n^5}{5} - \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \\
0^5 + 1^5 + 2^5 + 3^5 + \cdots + (n-1)^5 &= \frac{n^6}{6} - \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12} \\
0^6 + 1^6 + 2^6 + 3^6 + \cdots + (n-1)^6 &= \frac{n^7}{7} - \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42} \\
0^7 + 1^7 + 2^7 + 3^7 + \cdots + (n-1)^7 &= \frac{n^8}{8} - \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^4}{24} + \frac{n^2}{12} \\
0^8 + 1^8 + 2^8 + 3^8 + \cdots + (n-1)^8 &= \frac{n^9}{9} - \frac{n^8}{2} + \frac{2n^7}{3} - \frac{7n^5}{15} + \frac{2n^3}{9} - \frac{n}{30}
\end{aligned}$$

6 Problems

1. Show that
$$0^2 + 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. Let F_k be the Fibonacci numbers defined by: $F_0 = 0$, $F_1 = 1$, and if $k > 1$, $F_k = F_{k-1} + F_{k-2}$. Show that:
$$F_{n-1}F_{n+1} = F_n^2 + (-1)^n$$
and that
$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}.$$

3. Show that:
$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}.$$

4. Show that:
$$2! \cdot 4! \cdot 6! \cdots (2n)! \geq ((n+1)!)^n.$$

5. Show that:
$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} = 2\cos\frac{\pi}{2^{n+1}},$$
where there are n 2s in the expression on the left.

6. (Chebyshev Polynomials) Define $P_i(x)$ as follows:
$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_{n+1}(x) &= xP_n(x) - P_{n-1}(x), \text{for } n > 0. \end{aligned}$$
Show that
$$P_n(2\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

7. Show that:
$$\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta = \frac{\sin\left(\frac{(n+1)\theta}{2}\right)\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

8. (Quicksort) Prove the correctness of the following computer algorithm to sort a list of n numbers into ascending order. Assume that the original list is
$$\{x_0, x_1, \ldots, x_{n-1}\}.$$
Sort(j,k) where $j \leq k$ sorts the elements from x_j to x_{k-1}. In other words, to sort the entire list of n elements, call Sort(0, n). (Note that Sort(j, j) sorts an empty list.)

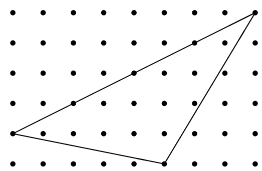
   Here is the algorithm:

   - (Case 1) If $k - j \leq 1$, do nothing.
   - (Case 2) If $k - j > 1$, rearrange the elements from x_{j+1} through x_{k-1} so that the first slots in the list are filled with numbers smaller than x_j, then put in x_j, and then all the numbers larger than x_j. (This can be done by running a pointer from the $(k-1)^{th}$ slot down and from the $(j+1)^{th}$ slot up, swapping elements that are out of order. Then put x_j into the slot between the two lists.) After this rearrangement, suppose that x_j winds up in slot m, where $j \leq m < k$. Now apply Sort(j,m) and Sort($m+1$, k).

9. (Towers of Hanoi) Suppose you have three posts and a stack of n disks, initially placed on one post with the largest disk on the bottom and each disk above it is smaller than the disk below. A legal move involves taking the top disk from one post and moving it so that it becomes the top disk on another post, but every move must place a disk either on an empty post, or on top of a disk larger than itself. Show that for every n there is a sequence of moves that will terminate with all the disks on a post different from the original one. How many moves are required for an initial stack of n disks?

10. (Pick's Theorem) Given a simple polygon in the plane whose vertices lie on lattice points, show that the area of the polygon is given by $I + B/2 - 1$, where I is the number of lattice points entirely within the polygon and B is the number of lattice points that lie on the boundary of the polygon.

    A simple polygon is a closed loop of line segments whose only points in common are the endpoints of adjacent pairs of segments. In other words, the edges of the polygon do not touch each other, except at the vertices, where exactly two edges meet. Note that a simple polygon need not be convex.

    In the example above, the triangle includes 6 boundary points and 12 interior points, so its area should be (according to Pick's Theorem) $12 + 6/2 - 1 = 14$. You can check that this is right by noticing that its area is the area of the surrounding rectange ($5 \cdot 8 = 40$) less the areas of the three surrounding triangles: (5/2, 15/2, and 32/2). When we check, we get: $40 - 5/2 - 15/2 - 32/2 = 14$.

11. (Arithmetic, Geometric, and Harmonic means) Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of positive numbers. We define the arithmetic, geometric, and harmonic means ($\mathcal{A}(A)$, $\mathcal{G}(A)$, and $\mathcal{H}(A)$, respectively) as follows:

$$\begin{aligned} \mathcal{A}(A) &= \frac{a_1 + a_2 + \cdots + a_n}{n} \\ \mathcal{G}(A) &= \sqrt[n]{a_1 a_2 \cdots a_n} \\ \mathcal{H}(A) &= \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \end{aligned}$$

    Show that

$$\mathcal{H}(A) \leq \mathcal{G}(A) \leq \mathcal{A}(A).$$

    In the solution section, the actual solution is preceeded by a couple of hints.

12. (Catalan numbers) Given n pairs of parentheses, Let T_n be the number of ways they can be arranged in a valid mathematical expression. For example, if $n = 3$, there are 5 ways to rearrange the parentheses:

$$((())), (())(), ()(()), (()()), ()()(),$$

    so $T_3 = 5$. Let $T_0 = 1$. Show that:

$$T_n = \frac{1}{n+1}\binom{2n}{n}.$$

    Hint: Show that:

$$T_{i+1} = T_i T_0 + T_{i-1} T_1 + \cdots + T_0 T_i.$$

7 Solutions

1. If $n = 0$ we trivially have:
$$0^2 = 0(1)(1)/6.$$
Assume that the equation is true for $n = k$:
$$0^2 + 1^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \tag{6}$$
¿From this, we want to show that:
$$\begin{aligned} 0^2 + 1^2 + \cdots + k^2 + (k+1)^2 &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6}. \end{aligned}$$
Begin with Equation 6 and add $(k+1)^2$ to both sides:
$$0^2 + 1^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2.$$
Just do some algebra, and the proof is complete:
$$0^2 + \cdots + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$
$$0^2 + \cdots + (k+1)^2 = \frac{(k+1)(2k^2 + k + 6k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}.$$

2. Part 1:
First check for $n = 1$:
$$F_0F_2 = 0 \cdot 2 = 0 = F_1^2 + (-1)^1 = 1 - 1 = 0.$$
If we assume it is true for $n = k$, we have:
$$F_{k-1}F_{k+1} = F_k^2 + (-1)^k. \tag{7}$$
¿From this, we need to show that the equality continues to hold for $n = k+1$. In other words, we need to show if we begin with Equation 7 we can show that:
$$F_kF_{k+2} = F_{k+1}^2 + (-1)^{k+1}.$$
Since $F_{k+2} = F_k + F_{k+1}$, the equation above is equivalent to:
$$F_k(F_k + F_{k+1}) = F_{k+1}^2 + (-1)^{k+1},$$
or to
$$F_k^2 + F_kF_{k+1} = F_{k+1}^2 + (-1)^{k+1}.$$
Substitute F_k^2 from the right-hand-side of Equation 7:
$$F_{k-1}F_{k+1} - (-1)^k + F_kF_{k+1} = F_{k+1}^2 + (-1)^{k+1},$$

or

$$F_{k+1}(F_{k-1}+F_k) = F_{k+1}^2 + (-1)^{k+1} + (-1)^k = F_{k+1}^2,$$

or

$$F_{k+1}^2 = F_{k+1}^2.$$

Part 2:

For $n = 0$:

$$\sum_{i=0}^{0} F_i^2 = F_0^2 = 0 = F_0 F_1 = 0 \cdot 1 = 0.$$

If it's true for $n = k$:

$$\sum_{i=0}^{k} F_i^2 = F_k F_{k+1} \tag{8}$$

we can add F_{k+1}^2 to both sides of Equation 8 to get:

$$\sum_{i=0}^{k+1} F_i^2 = F_k F_{k+1} + F_{k+1}^2 = F_{k+1}(F_k + F_{k+1}) = F_{k+1} F_{k+2}.$$

3. For $n = 1$ we need to show that:

$$1 \le 2\sqrt{1} = 2.$$

Assume the equation is true for $n = k$:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} \le 2\sqrt{k}.$$

To show that it is also true for $n = k+1$, add $1/\sqrt{k+1}$ to both sides:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \le 2\sqrt{k} + \frac{1}{\sqrt{k+1}}.$$

If we can show that

$$2\sqrt{k} + \frac{1}{\sqrt{k+1}} \le 2\sqrt{k+1}$$

then we are done. Multiply both sides by $\sqrt{k+1}$ and then square both sides to obtain:

$$4k(k+1) + 4\sqrt{k(k+1)} + 1 \le 4(k^2 + 2k + 1).$$

Rearrange:

$$4\sqrt{k(k+1)} \le 4k + 3,$$

and square both sides again:

$$16k^2 + 16k \le 16k^2 + 24k + 9,$$

which is obviously true.

4. First show it is true for $n = 1$:

$$2 = 2! \geq (2!)^1 = 2.$$

Now assume it is true for $n = k$:

$$2! \cdot 4! \cdot 6! \cdots (2k)! \geq ((k+1)!)^k. \tag{9}$$

If we multiply both sides of Equation 9 by $(2(k+1))!$, we obtain:

$$2! \cdot 4! \cdots (2k)!(2k+2)! \geq ((k+1)!)^k (2k+2)!$$

If we can show that the right hand side of the equation above is larger than $((k+2)!)^{k+1}$, we are done. Notice that the term $(2k+2)!$ on the right hand side can be written:

$$(2k+2)! = (2k+2)(2k+1)(2k)\cdots(k+3)(k+2)!$$

This consists of k terms, all geater than $k+2$, multiplied by $(k+2)!$, so

$$\begin{aligned} ((k+1)!^k(2k+2)! &> ((k+1)!)^k(k+2)^k(k+2)! \\ &= ((k+2)!)^k(k+2)! = ((k+2)!)^{k+1}. \end{aligned}$$

5. For $n = 1$ we have:

$$\sqrt{2} = 2\cos\frac{\pi}{2^2} = 2\cos\pi/4 = 2\sqrt{2}/2 = \sqrt{2}.$$

Now assume it's true for k nested square roots:

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} = 2\cos\frac{\pi}{2^{k+1}}.$$

If we add 2 to both sides and take the square root, the left hand side will now have $k+1$ nested square roots, and the right hand side will be:

$$\sqrt{2+2\cos\frac{\pi}{2^{k+1}}}. \tag{10}$$

We just need to show that the value above is equal to

$$2\cos\frac{\pi}{2^{k+2}}. \tag{11}$$

We know that for any angle θ we have:

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}. \tag{12}$$

Substitute $\pi/2^{k+2}$ for θ in equation 12 and we can show the equality of the expressions 10 and 11 above.

6. (Chebyshev Polynomials) First, let's show that the formula holds for *both* $n = 0$ and $n = 1$. (For this example, we must do the proof for the first two cases, because to get to the case $k+1$, we need to use the result for k and for $k-1$.)

   Case $n = 0$:
$$1 = P_0(2\cos\theta) = \frac{\sin\theta}{\sin\theta} = 1.$$
   Case $n = 1$:
$$2\cos\theta = P_1(2\cos\theta) = \frac{\sin 2\theta}{\sin\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta} = 2\cos\theta.$$
   Now assume that it's true for $n = k$ and $n = k-1$, where $k > 0$:
$$P_k(2\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta}, P_{k-1}(2\cos\theta) = \frac{\sin k\theta}{\sin\theta}.$$
   ¿From the definition of $P_{k+1}(x)$ we then have:
$$\begin{aligned} P_{k+1}(2\cos\theta) &= 2\cos\theta P_k(2\cos\theta) - P_{k-1}(2\cos\theta) \\ &= 2\cos\theta\frac{\sin(k+1)\theta}{\sin\theta} - \frac{\sin k\theta}{\sin\theta} \end{aligned}$$
   Use the trick that $k\theta = (k+1)\theta - \theta$ to rewrite the right hand side of the equation above as:
$$\begin{aligned} P_{k+1}(2\cos\theta) &= \frac{2\cos\theta\sin(k+1)\theta - \sin((k+1)\theta - \theta)}{\sin\theta} \\ &= \frac{2\cos\theta\sin(k+1)\theta - \cos\theta\sin(k+1)\theta + \cos(k+1)\theta\sin\theta}{\sin\theta} \\ &= \frac{\cos\theta\sin(k+1)\theta + \cos(k+1)\theta\sin\theta}{\sin\theta} \\ &= \frac{\sin(k+2)\theta}{\sin\theta}. \end{aligned}$$

7. To simplify the notation, let's let:
$$S_k(\theta) = \sin\theta + \sin 2\theta + \cdots + \sin k\theta.$$
   To prove the statement for $n = 1$ we need to check that:
$$\sin\theta = S_1(\theta) = \frac{\sin(2\theta/2)\sin(\theta/2)}{\sin(\theta/2)} = \sin\theta.$$
   Assume it is true for $n = k$:
$$S_k(\theta) = \frac{\sin\frac{(k+1)\theta}{2}\sin\frac{k\theta}{2}}{\sin\frac{\theta}{2}}.$$
   Since $S_{k+1}(\theta) = S_k(\theta) + \sin(k+1)\theta$, we have:
$$S_{k+1}(\theta) = \frac{\sin\frac{(k+1)\theta}{2}\sin\frac{k\theta}{2} + \sin(k+1)\theta\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$
   Now, using the fact that $\sin(k+1)\theta = \sin 2((k+1)\theta/2)$ and that for any angle γ, $\sin 2\gamma = 2\sin\gamma\cos\gamma$:

$$S_{k+1}(\theta) = \frac{\sin\frac{(k+1)\theta}{2}\sin\frac{k\theta}{2} + 2\sin\frac{(k+1)\theta}{2}\cos\frac{(k+1)\theta}{2}\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$S_{k+1}(\theta) = \frac{\sin\frac{(k+1)\theta}{2}\left(\sin\frac{k\theta}{2} + 2\cos\frac{(k+1)\theta}{2}\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}}$$

Now use the trick that $\sin(k\theta/2) = \sin((k+1)\theta/2 - \theta/2)$ and expand it as the sine of a sum of angles:

$$S_{k+1}(\theta) = \frac{\sin\frac{(k+1)\theta}{2}\left(\sin\frac{(k+1)\theta}{2}\cos\frac{\theta}{2} - \cos\frac{(k+1)\theta}{2}\sin\frac{\theta}{2} + 2\cos\frac{(k+1)\theta}{2}\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}}$$

$$S_{k+1}(\theta) = \frac{\sin\frac{(k+1)\theta}{2}\left(\sin\frac{(k+1)\theta}{2}\cos\frac{\theta}{2} + \cos\frac{(k+1)\theta}{2}\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}}$$

$$S_{k+1}(\theta) = \frac{\sin\frac{(k+1)\theta}{2}\sin\frac{(k+2)\theta}{2}}{\sin\frac{\theta}{2}}$$

8. (Quicksort) To show that the quicksort algorithm works, use induction (surprise!) First, we'll show that it works for sets of size zero or of size 1. Those sets are already sorted, so there is nothing to do, and since they fall under the first case of the algorithm which says to do nothing, we are in business.

   If the quicksort algorithm works for all sets of numbers of size k or smaller, then if we start with a list of size $k+1$, since we pick out one element for comparisons and divide the rest of the set into two subsets, it is obvious that each of the subsets has size smaller than or equal to k. Since the algorithm works on all of those, we know that the full algorithm works since the numbers smaller than the test number are sorted, then comes the test number, then comes a sorted list of all the numbers larger than it.

   This algorithm is heavily used in the real world. Surprisingly, the algorithm's performance is worst if the original set is already in order. Can you see why?

9. (Towers of Hanoi) Again, this is an easy induction proof. If there is only one disk on a post, you can immediately move it to another post and you are done.

   If you know that it is possible to move k disks to another post, then if you initially have $k+1$ disks, move the top k of them to a different post, and you know that this is possible. Then you can move the largest disk on the bottom to the other empty post, followed by a movement of the k disks to that other post.

   This method, which can be shown to be the fastest possible, requires $2^k - 1$ steps to move k disks. This can also be shown by induction—if $k = 1$, it requires $2^1 - 1 = 1$ move. If it's true for stacks of size up to k disks, then to move $k+1$ requires $2^k - 1$ (to move the top k to a different post) then 1 (to move the bottom disk), and finally $2^k - 1$ (to move the k disks back on top of the moved bottom). The total for $k+1$ disks is thus $(2^k - 1) + 1 + (2^k - 1) = 2 \cdot 2^k - 1 = 2^{k+1} - 1$.

   The above proof doesn't actually spell out an algorithm to solve the towers of Hanoi problem, but here is such an algorithm. You may be interested in trying to show that the following method always works:

Suppose the posts are numbered 1, 2, and 3, and the disks begin on post 1. Take the smallest disk and move it every other time. In other words, moves 1, 3, 5, 7, et cetera, are all of the top disk. Move the top disk in a cycle—first to post 2, then 3, then 1, then 2, then 3, then 1, ... On even moves, make the only possible move that does not involve the smallest disk. This will solve the problem.

10. (Pick's Theorem) The proof of this depends on the fact that an n-sided polygon, even one that is concave, can be divided into two smaller polygons by connecting two vertices together so that the connecting diagonal lies completely inside the polygon. This can obviously be continued until the original polygon is divided into triangles.

A 4-sided polygon (a quadrilateral) is thus split into two triangles; a 5-sided polygon into 3 triangles, et cetera, and in general, an n-sided polygon is split into $n - 2$ triangles.

We are going to prove Pick's theorem by induction on the number of sides of the polygon. We will start with $n = 3$, since the theorem makes sense only for polygons with three or more sides.

If we can show that it works for triangles then we've proven the theorem for the case $n = 3$. We then assume that it holds for all polygons with k or fewer edges, and from that, show that it works for polygons with $k + 1$ edges.

We'll delay the proof for $k = 3$ for a moment, and look at how to do the induction step. When your $k + 1$-sided polygon is split, it will be split into two smaller polygons that have an edge in common, and that both have k or fewer edges, so by the induction hypothesis, Pick's Theorem can be applied to both of them to calculate their areas based on the number of internal and boundary points. The area of the original polygon is the sum of the areas of the smaller ones.

Suppose the two sub-polygons of the original polygon P are P_1 and P_2, where P_1 has I_1 interior points and B_1 boundary points. P_2 has I_2 interior and B_2 boundary points. Let's also assume that the common diagonal of the original polygon between P_1 and P_2 contains m points. For concreteness, let's assume that P has I interior and B boundary points.

$$\mathcal{A}(P) = \mathcal{A}(P_1) + \mathcal{A}(P_2) = (I_1 + B_1/2 - 1) + (I_2 + B_2/2 - 1).$$

Since any point interior to P_1 or P_2 is interior to P, and since $m-2$ of the common boundary points of P_1 and P_2 are also interior to P, $I = I_1+I_2+m-2$. Similar reasoning gives $B = B_1+B_2-2(m-2)-2$. Therefore:

$$\begin{aligned} I + B/2 - 1 &= (I_1 + I_2 + m - 2) + (B_1 + B_2 - 2(m - 2) - 2)/2 - 1 \\ &= (I_1 + B_1/2 - 1) + (I_2 + B_2/2 - 1) = \mathcal{A}(P). \end{aligned}$$

The easiest way I know to show that Pick's Theorem works for triangles is to show first that it works for rectangles that are aligned with the lattice, then to show that it works for right triangles aligned with the lattice, and using that, we show that it works for arbitrary triangles.

For rectangles, it's easy. Suppose the rectangle R is of size n by m. There will be $2m + 2n$ boundary points and $(m - 1)(n - 1)$ interior points (convince yourself this is true with a drawing). Thus, $B = 2m + 2n$, $I = (m - 1)(n - 1)$, and the area is mn. So:

$$mn = \mathcal{A}(R) = I + B/2 - 1 = (m - 1)(n - 1) + m + n - 1 = mn.$$

Any right triangle T can be extended to a rectangle by placing a copy of it on the other side of its diagonal. If the triangle has sides m, n, and $\sqrt{m^2 + n^2}$, its area is $mn/2$. If there are k points on the diagonal, the number of interior points of the triangle is $((m-1)(n-1)-k)/2$. The number of boundary points is $m+n+1+k$. Check Pick's formula:

$$\begin{aligned} mn/2 &= \mathcal{A}(T) = I + B/2 - 1 \\ &= ((m-1)(n-1)-k)/2 + (m+n+1-k)/2 - 1 = mn/2. \end{aligned}$$

Any triangle that is not a right triangle can be surrounded by a rectangle and its area can be written as the area of the rectangle minus the areas of at most three right triangles. The proof for this final case is left as an easy exercise. The manipulations are very similar to those shown in the proofs above.

11. (Arithmetic, Geometric, and Harmonic means)

Hint: Once you prove that $\mathcal{G}(A) \leq \mathcal{A}(A)$, then you can show find a relationship between the harmonic and arithmetic means that proves the inequality between those two means.

Hint: Prove that $\mathcal{G}(A) \leq \mathcal{A}(A)$ if the set A has 2^n elements. Later, show it is true for an arbitrary number of elements.

Solution:

We will first show that

$$\mathcal{G}(A) \leq \mathcal{A}(A) \tag{13}$$

if A contains 2^n values. This can be done by induction. If $n = 0$, then Equation 13 amounts to:

$$a_1 \leq a_1$$

which is trivially true.

To see how the induction step works, just look at going from $n = 0$ to $n = 1$. We want to show that:

$$\sqrt{a_1 a_2} \leq \frac{a_1 + a_2}{2}.$$

Square both sides, so our problem is equivalent to showing that:

$$a_1 a_2 \leq \frac{a_1^2 + 2a_1a_2 + a_2^2}{4}$$

or that

$$0 \leq \frac{a_1^2 - 2a_1a_2 + a_2^2}{4} = \frac{(a_1 - a_2)^2}{4}.$$

This last result is clearly true, since the square of any number is positive.

So in general, suppose it's true for sets of size $k = 2^n$ and we need to show it's true for sets of size $2k = 2^{n+1}$, or in other words show that:

$$\sqrt[2k]{a_1 a_2 \cdots a_{2k}} \leq \frac{a_1 + a_2 + \cdots + a_{2k}}{2k}. \tag{14}$$

Rewrite Equation 14 as:

$$\sqrt{\sqrt[k]{a_1 \cdots a_k}\,\sqrt[k]{a_{k+1} \cdots a_{2k}}} \leq \frac{\dfrac{a_1 + \cdots + a_k}{k} + \dfrac{a_{k+1} + \cdots + a_{2k}}{k}}{2}.$$

If we let

$$\begin{aligned} a &= \sqrt[k]{a_1 a_2 \cdots a_k} \\ b &= \sqrt[k]{a_{k+1} a_{k+2} \cdots a_{2k}} \\ A &= \frac{a_1 + \cdots + a_k}{k} \\ B &= \frac{a_{k+1} + \cdots + a_{2k}}{k} \end{aligned}$$

and we know that $a < A$ and $b < B$ (because the induction hypothesis tells us so for $k = 2^n$) then we need to show that

$$\sqrt{ab} \leq \frac{A+B}{2}.$$

But we showed above that $\sqrt{AB} \leq (A+B)/2$, and we know that $\sqrt{ab} \leq \sqrt{AB}$ so we are done.

But of course, not all sets have a cardinality that is exactly a power of 2. Suppose we want to show that it's true for a set of cardinality m, where $m < k = 2^n$.

Our set $A = \{a_1, a_2, \ldots, a_m\}$ contains m elements. Let

$$u = \frac{a_1 + a_2 + \cdots + a_m}{m}.$$

If we add $m - k$ copies of u to the original members of the set A, we will have a new set A' with $k = 2^n$ members: $A' = \{a_1, a_2, \ldots, a_m, u, u, ..., u\}$. Since we know that $\mathcal{G}(A') \leq \mathcal{A}(A')$, we have:

$$\sqrt[k]{a_1 \cdots a_m u^{k-m}} \leq \frac{a_1 + a_2 + \cdots + a_m + (k-m)u}{k}. \tag{15}$$

If we raise both sides of Equation 15 to the power k and do some algebra, we get:

$$a_1 a_2 \cdots a_m u^{k-m} \leq \Big(\frac{a_1 + a_2 + \cdots + a_m + (k-m)u}{k}\Big)^k$$

$$a_1 a_2 \cdots a_m \leq \Big(\big(\frac{m}{k}\big)\big(\frac{a_1 + \cdots + a_m}{m}\big) + \big(\frac{k-m}{k}\big)\big(\frac{a_1 + \cdots + a_m}{m}\big)\Big)^k u^{m-k}.$$

$$a_1 a_2 \cdots a_m \leq u^k u^{m-k} = u^m = \big(\frac{a_1 + a_2 + \cdots + a_m}{m}\big)^m,$$

which is exactly what we were trying to prove.

Now to complete the problem, we need only show that $\mathcal{H}(A) \leq \mathcal{G}(A)$. To show this, consider the set $A' = \{1/a_1, 1/a_2, \ldots 1/a_m\}$.

We know that the geometric mean is less than the arithmetic mean, so apply that fact to the set A':

$$\frac{1}{a_1 a_2 \cdots a_m} \leq \frac{\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_m}\right)}{m}.$$

If we invert both sides (which will flip the direction of the inequality), we have the desired result.

12. (Catalan numbers) We will begin by showing that

$$T_{i+1} = T_i T_0 + T_{i-1} T_1 + \cdots + T_0 T_i. \tag{16}$$

To see this, begin with the leftmost matched pair of parentheses. It can contain between zero and i matched pairs inside it. If it contains k matched pairs, the remaining parentheses on the right contain $i - k$ pairs. The k matched pairs can be arranged in T_k ways, and the remainder in T_{i-k} ways, for $k = 0, \ldots, i$. So expression 16 holds.

Let

$$f(x) = \sum_{k=0}^{\infty} T_k x^k = T_0 + T_1 x + T_2 x^2 + \cdots$$

$$\begin{aligned}[f(x)]^2 &= T_0T_0 + (T_1T_0 + T_0T_1)x + (T_2T_0 + T_1T_1 + T_0T_2)x^2 + \cdots \\ &= T_1 + T_2x + T_3x^2 + \cdots\end{aligned}$$

$$T_0 + x[f(x)]^2 = T_0 + T_1x + T_2x^2 + \cdots = f(x),$$

and, since $T_0 = 1$:

$$x[f(x)]^2 - f(x) + 1 = 0.$$

Now use the quadratic formula to solve for $f(x)$:

$$f(x) = \frac{1 \pm (1-4x)^{1/2}}{2x}. \tag{17}$$

The binomial theorem states that:

$$(1+u)^n = 1 + \binom{n}{1}u + \binom{n}{2}u^2 + \binom{n}{3}u^3 + \cdots,$$

so if $n = 1/2$ in eqation 17, we have:

$$(1-4x)^{1/2} = 1 - \binom{\frac{1}{2}}{1}(4x) + \binom{\frac{1}{2}}{2}(4x)^2 - \binom{\frac{1}{2}}{3}(4x)^3 + \cdots$$

In order for equation 17 to make sense, we need the negative value of the $\pm$, and we obtain:

$$f(x) = \binom{\frac{1}{2}}{1}\frac{4x}{2x} - \binom{\frac{1}{2}}{2}\frac{(4x)^2}{2x} + \binom{\frac{1}{2}}{3}\frac{(4x)^3}{2x} - \cdots = \sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}\frac{(-4x)^k}{2x}.$$

Shifting the index k, we obtain:

$$f(x) = \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k+1}\frac{(-4)^{k+1}x^k}{2}. \tag{18}$$

T_k will be the coefficient of x^k in equation 18:

$$T_k = \binom{\frac{1}{2}}{k+1}\frac{(-4)^{k+1}x^k}{2} = \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2k-1}{2}4^{k+1}k^k}{2(k+1)!}$$

$$T_k = \frac{(2k-1)!!4^{k+1}}{(k+1)!\cdot 2\cdot 2^{k+1}},$$

where $u!! = u(u-2)(u-4)\cdots 1$. $(2k-1)!!$ can be multiplied by $2^k k!$ to yield $(2k)!$ so we have:

$$\begin{aligned}T_k &= \frac{(2k)!4^{k+1}}{2^k\cdot 2\cdot 2^{k+1}k!(k+1)!} \\ &= \frac{1}{k+1}\frac{(2k)!}{k!k!} = \frac{1}{k+1}\binom{2k}{k}.\end{aligned}$$

Catalan numbers come up in a huge number of examples. For example, given a regular polygon with n sides, the Catalan numbers count the number of ways that the polygon can be uniquely triangulated. To be precise, if a polygon has $n+2$ sides, the number of ways to triangulate it is given by:

$$\frac{1}{n+1}\binom{2n}{n}.$$