# Incidence is no Coincidence
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We've been pretending to travel over various train systems for the past few weeks, partly as an excuse to consider interesting counting problems. We will discover soon that the underlying mathematical structure of the train system "graph" (stops = vertices, tracks between adjacent stops = edges) has many interesting applications unrelated to train travel. We will still want to compute quantities such the number of paths of length "n" between different vertices, and to do so more systematically than we have done up until now. It turns out that there is an amazing computational and theoretical connection between graphs and matrix theory, which we will now explore.

Noville:
Consider the train system indicated by the directed graph below. Notice that the tracks are only one-way. The mathematical term for such a collection of vertices and directed edges is a directed graph, or digraph. If all tracks run in both directions the corresponding mathematical structure is called a (plain) graph. We will be more precise later.
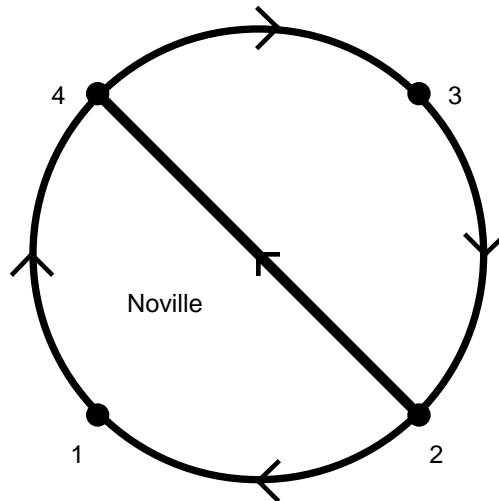
(1)   Consider trips which start at station 2, and tabulate trips of length up to 10. Organize the table today so that the vertices are arranged vertically and the trip lengths increase horizontally:

<-- Trip length -->

	0	1	2	3	4	5	6	7	8	9	10	11
1	0	1	0	0	1	1	0	1	2	1	1	
2	1	0	0	1	1	0	1	2	1	1	3	
3	0	0	1	1	0	1	2	1	1	3	3	
4	0	1	1	0	1	2	1	1	3	3	2	

This tabulation process is becoming tedious. And suppose we wanted to record all trips from all stations! Luckily for us, if you can compute matrix powers it turns out to be easy to let your calculator track all this data.

Begin with the matrix which in row i of column j records the trips of length 1 which originate at vertex j and end at vertex i. In other words, there is a 1 in this entry if there is an allowed edge from j to i, and otherwise there is a 0. This matrix is called the incidence matrix of the digraph (or graph).

(2) Can you construct the incidence matrix for Noville?

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

If you start multiplying the incidence matrix by itself, in other words taking powers of A, you may notice an interesting pattern:

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad A^{10} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 2 & 3 & 3 \end{bmatrix}$$

The second columns seem to reproduce our trip table. So probably the other three columns tell us similar information. Assuming they do, you should be able to answer questions like:

(3) How many trips of length 10 start at station 3 and end at station 2? Is this the same as the number which start at station 2 and end at station 3? How many trips of length 10 start and end at the same vertex?

Can it be a coincidence that the matrix powers of the incidence matrix give the trip tables? Of course not!

Theorem: The ij entry in the nth power of the incidence matrix for any graph or digraph is exactly the number of different paths of length n, beginning at vertex j and ending at vertex i.

We can illustrate the theorem by doing a computation. From this example you can see how to prove the general result by induction.

(4) Let's assume that the 10th power of A does reproduce the trip tables for Noville correctly. Let's compute the 11th power, and verify that it must then reproduce the table for trips of length 11.

$$A^{11} = [A] [A^{10}]$$

$$A^{11} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 2 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 & 1 & 2 & 3 & 3 \end{bmatrix}$$

Remember, to get the ij entry in the product AB you focus on row i of A and column j of B, multiply the corresponding entries together, and add up. For example, to get the entry in row 4 and column 2 of the product we focus on the fourth row of A and the second column of B, in our case:

$$[1 \quad 1 \quad 0 \quad 0] \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

We then compute

$$1 \cdot 1 + 1 \cdot 3 + 0 \cdot 3 + 0 \cdot 2 = 4$$

What does this computation mean in terms of trips of length 11, starting at 2 and ending at 4? All such paths can be constructed from paths of length 10 starting at 2, by adding a final segment. According to first entry in the second column of A^{10}, there is 1 length-10 path from 2 to 1 According to first entry in the 4th row of A, we can append a

final segment to this trip by following the edge from 1 to 4, giving a path of length 11 from 2 to 4. Similarly, there are 3 length-10 paths from 2 to 2, and there is an edge from 2 to 4. There are 3 long paths from 2 to 3, but no edge from 3 to 4. There are 2 long paths from 2 to 4, but no edge from 4 to itself. Thus the total number of length 11 trips from vertex 2 to vertex 4 is exactly the computation we did in the matrix product entry:

$$1 \cdot 1 + 1 \cdot 3 + 0 \cdot 3 + 0 \cdot 2 = 4$$

Thus if A^{10} gives the correct trip table for paths of length 10, it follows by considering each entry of its product with A, that A^{11} is correct for trips of length 11!

$$A^{11} = \begin{matrix} 2 & 3 & 1 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 2 & 3 & 3 \\ 3 & 4 & 2 & 3 \end{matrix}$$

(5) How many trips of length 100 start at station 1 and end at station 2? (Perhaps you wish to use your calculator here!)

$$A^{100} = \begin{matrix} 59993817 & 109139117 & 89404037 & 73237462 \\ 73237462 & 133231279 & 109139117 & 89404037 \\ 89404037 & 162641499 & 133231279 & 109139117 \\ 109139117 & 198543154 & 162641499 & 133231279 \end{matrix}$$

---so the answer is 73,237,462.

(6) What graph or digraph has the following incidence matrix? Have we been there before?

$$B := \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$