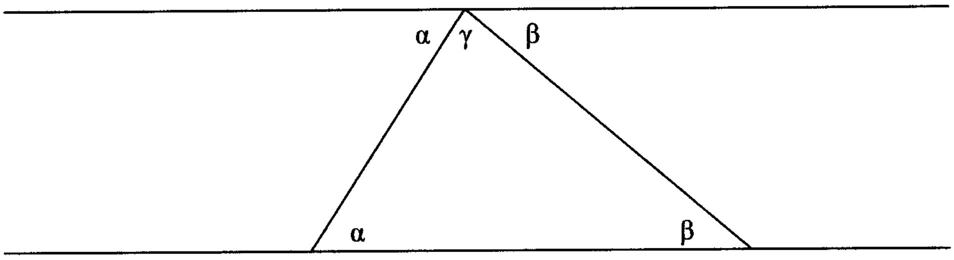
Solutions

1.



2. Use induction, so suppose that the formula is true for $n = k$. (We know that the formula holds in the first case, $n = 3$.) Since a polygon with $k+1$ vertices is a polygon with k vertices with a triangle attached, the sum of its interior angles is $180(k-2) + 180 = 180(k-1) = 180((k+1) - 2)$.

3. Let the angles be x, $x+20$, $x+40$, $x+60$, and $x+80$. Adding these angles, we must have $5x + 200 = 180(5-2) = 540$. Thus, $5x = 340$, so $x = 68$. The middle angle, $x+40$, is the average of all 5 angles; thus, the average is $68+40 = 108$. Note, however, that the average of the interior angles of a pentagon is always 108!!

1. (C) Let d be the common difference of the arithmetic sequence. Then

$$75 + (75+d) + (75+2d) + (75+3d) = 360$$

$$300 + 6d = 360,$$

so $d = 10$ and the number of degrees in the largest angle is $75 + 3d = 105$.

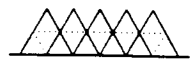
OR

Since the opposite bases of a trapezoid are parallel, the smallest and largest angles must be supplementary. Consequently, the answer is $180° - 75° = 105°$.

Note. If four angles in an arithmetic sequence sum to $360°$, then the smallest and largest must be supplementary, as must the two intermediate sized angles. Therefore, any quadrilateral whose consecutive angles form an arithmetic sequence is a trapezoid. Any isosceles trapezoid shows that the converse is not true.

2. (E) Partition the figure into 16 equilateral triangles as shown. Since each side of each of these 16 triangles has length

$$s = \frac{1}{2}\left(2\sqrt{3}\right) = \sqrt{3},$$



the total area is

$$16\left(s^2\frac{\sqrt{3}}{4}\right) = 16\left(3\frac{\sqrt{3}}{4}\right) = 12\sqrt{3}.$$

OR

In each of the two equilateral triangles at the ends, insert the segment connecting the midpoints of the outer side and the side on the line, as indicated. Observe that the area covered is enclosed by 5 congruent rhombi and 6 congruent equilateral triangles. The rhombi have diagonals with lengths 3 and $\sqrt{3}$, while the triangles have sides $\sqrt{3}$. Hence the area is

$$5\left(\frac{3\sqrt{3}}{2}\right) + 6\left(3\frac{\sqrt{3}}{4}\right) = 12\sqrt{3}.$$

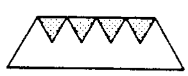
OR

Since the five equilateral triangles overlap in four smaller triangles whose sides are half as large, the area is

$$5\left(12\frac{\sqrt{3}}{4}\right) - 4\left(3\frac{\sqrt{3}}{4}\right) = 12\sqrt{3}.$$

OR

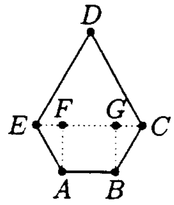
The area covered consists of a trapezoid with height 3 and bases $6\sqrt{3}$ and $4\sqrt{3}$ minus four equilateral triangles whose sides and altitudes are half those of the given triangles. Therefore the answer is



$$3 \cdot \frac{6\sqrt{3} + 4\sqrt{3}}{2} - 4\left(3\frac{\sqrt{3}}{4}\right) = 12\sqrt{3}.$$

3. **(C) The large circle has radius 3, so its area is $\pi \cdot 3^2 = 9\pi$. The seven small circles have a total area of $7\left(\pi \cdot 1^2\right) = 7\pi$. So the shaded region has area $9\pi - 7\pi = 2\pi$.**

4 **(B)** Draw $\overline{CE}$. Since $EA = BC$ and $\angle A = \angle B$, it follows that $ABCE$ is an isosceles trapezoid. Let F be the foot of the perpendicular from A to $\overline{CE}$, and G be the foot of the perpendicular from B to $\overline{CE}$. Then $EF = CG$. Since $\angle GBC = 30°$, we have



$$CG = \frac{1}{2}(BC) = 1 \text{ and } BG = \frac{\sqrt{3}}{2}(BC) = \sqrt{3}.$$

Now

$$CE = CG + GF + FE = 1 + 2 + 1 = 4,$$

so CDE is an equilateral triangle. Thus,

$$[ABCE] = \frac{1}{2}(BG)(AB + CE) = \frac{1}{2}\sqrt{3}\,(2+4) = 3\sqrt{3},$$

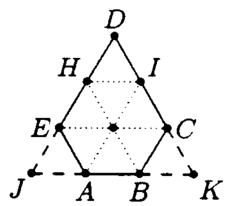
$$\text{and } [CDE] = \frac{\sqrt{3}}{4}(CE)^2 = \frac{\sqrt{3}}{4}(16) = 4\sqrt{3}.$$

Therefore,

$$[ABCDE] = [ABCE] + [CDE] = 7\sqrt{3}.$$

OR

Draw $\overline{HI}$ where H is the midpoint of $\overline{ED}$ and I is the midpoint of $\overline{CD}$. Then $ABCIHE$ is a regular hexagon and $\triangle HDI$ is congruent to any of the six equilateral triangles of side 2 that make up $ABCIHE$. Thus, the area of $ABCDE$ is the sum of the areas of 7 equilateral triangles of side 2, so it is $7\left(2^2\frac{\sqrt{3}}{4}\right) = 7\sqrt{3}$.



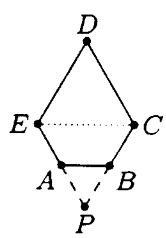
Note. A glance at the diagram for the previous solution shows that the area of $ABCDE$ is tiled by 7 of the 9 congruent equilateral triangles that tile equilateral triangle DJK. Since $JK = 3 \cdot 2 = 6$, the area of $ABCDE$ can by computed as

$$\frac{7}{9}\left(6^2\frac{\sqrt{3}}{4}\right) = 7\sqrt{3}.$$

OR

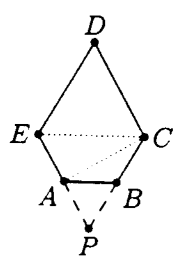
Extend $\overline{EA}$ and $\overline{CB}$ to meet at P. Since $\angle ABP = \angle BAP = 60°$, $\triangle ABP$ is equilateral as is $\triangle ECP$, and since $EC = CP = CB + BP = 4 = CD = DE$, $\triangle ECD$ is also equilateral. Now,

$$\begin{aligned}[ABCDE] &= ([ECD] + [ECP]) - [ABP] \\ &= 2\left(\frac{\sqrt{3}}{4}4^2\right) - \frac{\sqrt{3}}{4}2^2 = 7\sqrt{3}.\end{aligned}$$



OR

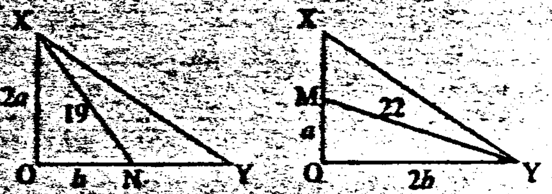
Construct P, and note that $\triangle PAB$ and $\triangle PEC$ are both equilateral, as above. Thus, $\angle AEC = 60°$. Because $\triangle ABC$ is isosceles, $\angle BAC = 30°$, so $\angle CAE = 90°$. Therefore, $\triangle CEA$ is a 30°-60°-90° triangle, so $CE = 4$, $AC = 2\sqrt{3}$ and the altitude from B to $\overline{AC}$ has length 1. Finally, $\triangle CDE$ is equilateral since $CE = 4 = CD = DE$. Hence,



$$\begin{aligned}[ABCDE] &= [ABC] + [CEA] + [CDE] \\ &= \frac{1}{2}\cdot 2\sqrt{3}\cdot 1 + \frac{1}{2}\cdot 2\cdot 2\sqrt{3} + \frac{\sqrt{3}}{4}\cdot 4^2 = 7\sqrt{3}.\end{aligned}$$

5 (B) Let $OM = a$ and $ON = b$. Then

$$19^2 = (2a)^2 + b^2 \quad \text{and} \quad 22^2 = a^2 + (2b)^2.$$



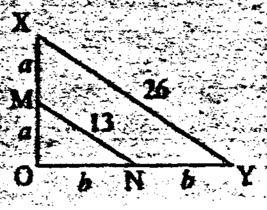
Hence

$$5(a^2 + b^2) = 19^2 + 22^2 = 845.$$

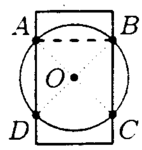
It follows that

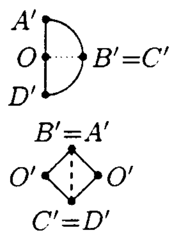
$$MN = \sqrt{a^2 + b^2} = \sqrt{169} = 13.$$

Since ΔXOY is similar to ΔMON and $XO = 2 \cdot MO$, we have $XY = 2 \cdot MN = 26$.



6 (D) Let O be the center of the circle and rectangle, and let the circle and rectangle intersect at A, B, C and D as shown. Since $AO = OB = 2$ and the width of the rectangle $AB = 2\sqrt{2}$, it follows that $\angle AOB = 90°$. Hence $\angle AOD = \angle DOC = \angle COB = 90°$. The sum of the areas of sectors AOB and DOC is $2\left(\frac{1}{4}\left(\pi 2^2\right)\right) = 2\pi$. The sum of the areas of isosceles right triangles AOD and COB is $2\left(\frac{1}{2} \cdot 2^2\right) = 4$. Thus, the area of the region common to both the rectangle and the circle is $2\pi + 4$.





Note. The sketch shows that sectors AOB and COD can be arranged to form a semicircle with area $\frac{1}{2}\pi 2^2 = 2\pi$ and the isosceles right triangles can be arranged to form a square with area $2^2 = 4$.

Comment. This problem can be more difficult if the radius and dimensions of the rectangle are chosen in a more arbitrary manner.

8 (E) Apply the *Law of Cosines* to $\triangle BAC$:

$$BC^2 = BA^2 + AC^2 - 2(BA)(AC)\cos A$$

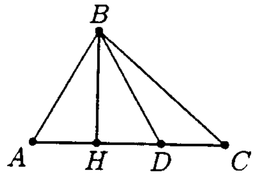
$$49 = 25 + 81 - 2(5)(9)\cos A.$$

Thus $\cos A = 19/30$. Let H be the foot of the altitude from B. Then

$$AD = 2 \cdot AH = 2(AB)\cos A = \frac{19}{3},$$

$$DC = AC - AD = 8/3, \text{ and}$$

$$AD : DC = 19 : 8.$$



OR

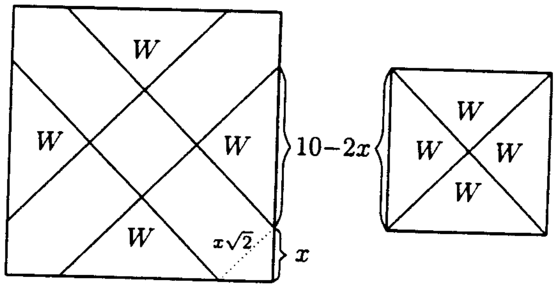
Apply the *Pythagorean Theorem* to triangles AHB and CHB:

$$5^2 - AH^2 = BH^2 = 7^2 - (9 - AH)^2.$$

Solve to find that $AH = 19/6$ so $AD = 2 \cdot AH = 19/3$. Thus

$$AD : DC = AD : (9 - AD) = \frac{19}{3} : \frac{8}{3} = 19 : 8.$$

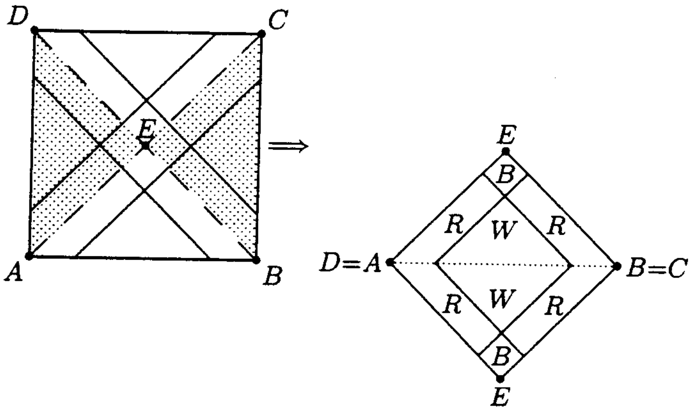
9 (C) Let the total area be 100 and let each red segment on the border of the flag be of length x. Then the four white triangles can be placed together to form a white square of area $100-36 = 64$ and side $10-2x$.



Since $(10-2x)^2 = 64$, we have $x = 1$. Thus the blue area is $(x\sqrt{2})^2 = 2$, which is 2% of the total area.

OR

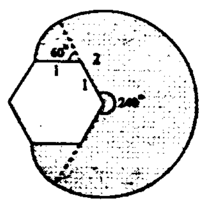
First note that the flag can be cut into four congruent isosceles right triangles by the two diagonals of the flag and that the percent of red, white, and blue areas in each of these triangles is the same as that in the flag. Then form a square translating $\triangle DCE$ down and attaching it to $\triangle ABE$ along their hypotenuses $\overline{AB}$ and $\overline{DC}$ as shown.



For simplicity assume that this "half-flag" is a 10×10 square. The interior white square consists of 64% of the area, so it must be 8×8. Thus the two blue squares measure 1×1, so they constitute 2% of the area.

10 **(E) Spot can go anywhere in a 240° sector of radius two yards and can cover a 60° sector of radius one yard around each of the adjoining corners. The total area is**

$$\pi(2)^2 \cdot \frac{240}{360} + 2\left(\pi(1)^2 \cdot \frac{60}{360}\right) = 3\pi.$$



11 (D) Let d be the common difference of the arithmetic sequence. The sum of the degrees in the interior angles of this hexagon is

$$m + (m-d) + (m-2d) + \cdots + (m-5d) = 6m - 15d.$$

The sum of the interior angles of any hexagon is $(6-2)180° = 720°$. Hence,

$$6m - 15d = 720 \qquad \text{or} \qquad 6m = 15d + 720 = 5(3d + 144),$$

so m is divisible by 5. Because the hexagon is convex, it follows that $m < 180$, and hence $m \le 175$. Since

$$65 + 87 + 109 + 131 + 153 + 175 = 720,$$

there is such a hexagon and $m° = 175°$.

OR

Let a be the smallest angle and d be the common difference of the arithmetic sequence. Then, the sum of the degrees in all six interior angles is

$$a + (a+d) + (a+2d) + \cdots + (a+5d) = 6a + 15d,$$

so $6a + 15d = 720$ and thus $2a + 5d = 240$. But the sum of the smallest and largest angles in the hexagon is

$$a + (a + 5d) = 2a + 5d = 240.$$

Since $2a = 240 - 5d = 5(48 - d)$, a must be divisible by 5, and hence the smallest and largest angles in the hexagon are divisible by 5. Therefore, the largest candidate for an interior angle is 175°, and we verify that there is a hexagon with this interior angle as in the previous solution.

OR

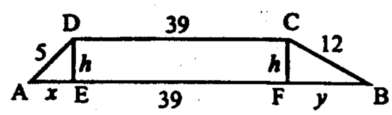
Derive $2a + 5d = 240$ as above. Then note that the solutions to this equation are all of the form $(a, d) = (120 - 5k, 2k)$ for k some integer. Since the hexagon is convex, the largest angle,

$$a + 5d = (2a+5d) - a = 240 - (120 - 5k) = 120 + 5k < 180.$$

The largest k that satisfies this inequality is $k = 11$, and it produces $a + 5d = 120 + 55 = 175$ as a candidate for the largest interior angle. Verify this candidate as in the first solution.

12 **(C)** First drop perpendiculars from D and C to $\overline{AB}$. Let E and F be the feet of the perpendiculars to $\overline{AB}$ from D and C, respectively, and let

$$h = DE = CF, \quad x = AE, \quad \text{and} \quad y = FB.$$



Then

$$25 = h^2 + x^2, \quad 144 = h^2 + y^2, \quad \text{and} \quad 13 = x + y.$$

So

$$144 = h^2 + y^2 = h^2 + (13 - x)^2 = h^2 + x^2 + 169 - 26x = 25 + 169 - 26x,$$

which gives $x = 50/26 = 25/13$, and

$$h = \sqrt{5^2 - \left(\frac{25}{13}\right)^2} = 5\sqrt{1 - \frac{25}{169}} = 5\sqrt{\frac{144}{169}} = \frac{60}{13}.$$

Hence

$$\text{Area }(ABCD) = \frac{1}{2}(39 + 52) \cdot \frac{60}{13} = 210.$$

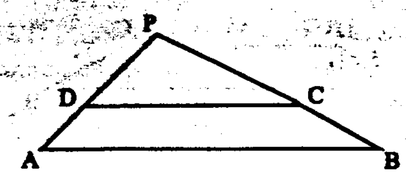
OR

Extend $\overline{AD}$ and $\overline{BC}$ to intersect at P. Since ΔPDC and ΔPAB are similar, we have

$$\frac{PD}{PD+5} = \frac{39}{52} = \frac{PC}{PC+12}.$$

So $PD = 15$ and $PC = 36$. Note that 15, 36, and 39 are three times 5, 12, and 13, respectively, so $\angle APB$ is a right angle. The area of the trapezoid is the difference of the areas of ΔPAB and ΔPDC, so

$$\text{Area}(ABCD) = \frac{1}{2}(20)(48) - \frac{1}{2}(15)(36) = 210.$$



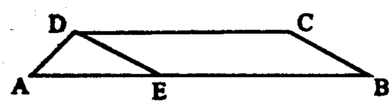
OR

Draw the line through D parallel to $\overline{BC}$, intersecting $\overline{AB}$ at E. Then $BCDE$ is a parallelogram, so $DE = 12$, $EB = 39$, and $AE = 52 - 39 = 13$. Thus $DE^2 + AD^2 = AE^2$, and ΔADE is a right triangle. Let h be the altitude from D to $\overline{AE}$, and note that

$$\text{Area}(ADE) = \frac{1}{2}(5)(12) = \frac{1}{2}(13)(h),$$

so $h = 60/13$. Thus

$$\text{Area}(ABCD) = \frac{60}{13} \cdot \frac{1}{2}(39 + 52) = 210.$$



7. Since the area of the big triangle is 6, the perpendicular to its hypotenuse has length 12/5. The shaded region is similar to the 3-4-5 triangle, with a similarity ratio of 3:5, so its area is 9/25 that of the big triangle, i.e., (9/25)(6) = 54/25.

13. We have two pairs of similar right triangles: ΔABW is similar to ΔHOW, and ΔABO is similar to ΔTWO. Thus, $\frac{AB}{OH} = \frac{BW}{OW}$, and $\frac{AB}{TW} = \frac{BO}{OW}$. Adding these two equations yields $AB(\frac{1}{OH} + \frac{1}{TW}) = \frac{BW + BO}{OW} = 1$. Solving for AB, we get

$$AB = \frac{1}{\frac{1}{L} + \frac{1}{R}} = \frac{LR}{L + R}.$$

OR

Let O be the origin in the xy-plane, and let S be the length of OW. Note that the length of AB is the y-coordinate of the intersection of the lines OT and HW. The line through O and T has the equation $y = \frac{R}{S}x$, and the equation of the line through H and W is $y = -\frac{L}{S}x + L$. Equating these and solving for x gives $x = \frac{SL}{R+L}$, so $y = \frac{RL}{L + R}$ as before.

14. You only need to cut triangle DAC into 2 pieces. Let E be the midpoint of AC. Since D and E are midpoints of AB and AC, respectively, DE is parallel to BC. Consequently, triangles ADE and ABC are similar, with a 1:2 similarity ratio. We can therefore put triangle DEC onto triangle DFC and triangle DEA onto triangle DFB to obtain triangle DBC from our two pieces.