MATH CIRCLE-PART 2

TOPIC: Unique Factorization
APPLICATION OF UNIQUE FACTORIZATION

We will see how unique factorization can be used to determine probabilities of rolls of dice. We seek to answer the following question:

Are there any other labelings of two six-sided dice that will give the same outcomes as two regular dice? I.e., instead of $1, \ldots, 6$ can we use other numbers or frequencies of numbers to get the same probabilities? *What is your educated guess?*

Next, let us see if any other labelings will give us these same probabilities. To begin with, we ask ourselves how we may obtain a sum of 6 with an ordinary pair of dice. There are 5 possibilities for the two faces: (1,5), (2,4), (3,3), (4,2), (5,1). Secondly, we consider the product of the two polynomials created by using the ordinary dice labels as exponents:

$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)$$

Observe that we pick up the term x^6 in this product in precisely the following ways: $x \cdot x^5$, $x^2 \cdot x^4$, $x^3 \cdot x^3$, $x^4 \cdot x^2$, and $x^5 \cdot x$. Notice the correspondence between pairs of labels whose sums are 6 and pairs of terms whose products are x^6. This correspondence is *one-to-one*, meaning that for every pair there is one and only one product whose exponents match it. Moreover, this correspondence is valid for all sums and all dice that yield the desired probabilities.

We will let $a_1, a_2, a_3, a_4, a_5, a_6$ and $b_1, b_2, b_3, b_4, b_5, b_6$ be any two lists of positive integer labels for a pair of cubes with the property that the probability of rolling any particular sum with these dice (let us call them *weird dice*) is the same as the probability of rolling that sum with ordinary dice labeled 1 through 6.

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Using our observation about products of polynomials, this means that:

(†) $(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x) =$

$(x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}).$

Now all we have to do is solve this equation for the a's and b's. Here is where unique factorization in $\mathbb{Z}[X]$ comes in. The polynomial $x^6 + x^5 + x^4 + x^3 + x^2 + x$ factors uniquely into irreducibles as: $x(x+1)(x^2+x+1)(x^2-x+1)$.

Consequently, the left-hand side of equation (†) has the irreducible factorization: $x^2(x+1)^2(x^2+x+1)^2(x^2-x+1)^2$.

Let $P(x) = (x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})$. Then by **the unique factorization** in $\mathbb{Z}[x]$, the only possible factors of $P(x)$ are

$$x, (x+1), (x^2+x+1), \text{ and } (x^2-x+1).$$

This means that $P(x)$ has the form: $x^r(x+1)^s(x^2+x+1)^t(x^2-x+1)^u$, where $0 \leq r, s, t, u \leq 2$.

We can make further restrictions. $P(1) = (1^{a_1} + 1^{a_2} + 1^{a_3} + 1^{a_4} + 1^{a_5} + 1^{a_6}) = 6$. On the other hand, $P(1) = 1^r \cdot 2^s \cdot 3^t \cdot 1^u$. As a result, $s = t = 1$.

Now consider r. $P(0) = (0^{a_1} + 0^{a_2} + 0^{a_3} + 0^{a_4} + 0^{a_5} + 0^{a_6}) = 0$. On the other hand, $P(0) = 0^r \cdot 1^s \cdot 1^t \cdot 1^u$. If $r = 0$, then $0^0 = 1$ and hence $P(0) = 1$, a contradiction. Therefore, $r > 0$. Suppose $r = 2$. Then the smallest possible sum one could roll with the corresponding labels for dice would be 3. (Note that we would have an x^2 in $P(x)$ as the smallest exponent, and an x in $Q(x) = x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}$ as the smallest possible exponent, since $Q(x)$ must have positive powers of x's. Thus, the smallest product of $P(x)Q(x)$ is x^3; i.e., the x^2 in $P(x)$ times x in $Q(x)$.) But 2 is the smallest power, using the LHS of (†). Therefore, $r \neq 2$.

Thus, we have reduced our possibilities to $r = 1, s = 1, t = 1$, and $u = 0, 1, 2$. We consider each case:

When $u = 0$, $P(x) = x^1(x+1)^1(x^2+x+1)^1 = x^4 + x^3 + x^3 + x^2 + x^2 + x$, so the die labels are 4, 3, 3, 2, 2, 1, a *weird die*. Moreover, by equation (†) and unique factorization, $Q(x) = x^1(x+1)^1(x^2+x+1)^1(x^2-x+1)^2 = x^8 + x^6 + x^5 + x^4 + x^3 + x$. Thus, the die labels corresponding to $Q(x)$ are 8,6,5,4,3,1. This is the other *weird die*.

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When $u = 1$, $P(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x$, so the die labels are 6, 5, 4, 3, 2, 1, an ordinary die. Since the factorization of this polynomial is $x(x+1)(x^2 + x + 1)(x^2 - x + 1)$, using (†), we determine that the other die is also an ordinary one.

When $u = 2$, $P(x) = x^8 + x^6 + x^5 + x^4 + x^3 + x$, so the die labels are 8, 6, 5, 4, 3, 1, the other *weird die*. This is essentially the same case as $u = 0$; i.e., the other die paired with this one is the $u = 0$ die.

This shows that the weird dice we considered, called *Sicherman dice*, DO give the same probabilities as ordinary dice AND that they are the ONLY other pair of dice that have this property.