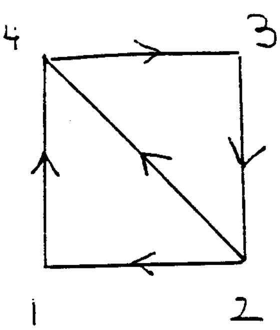
Directed Graphs

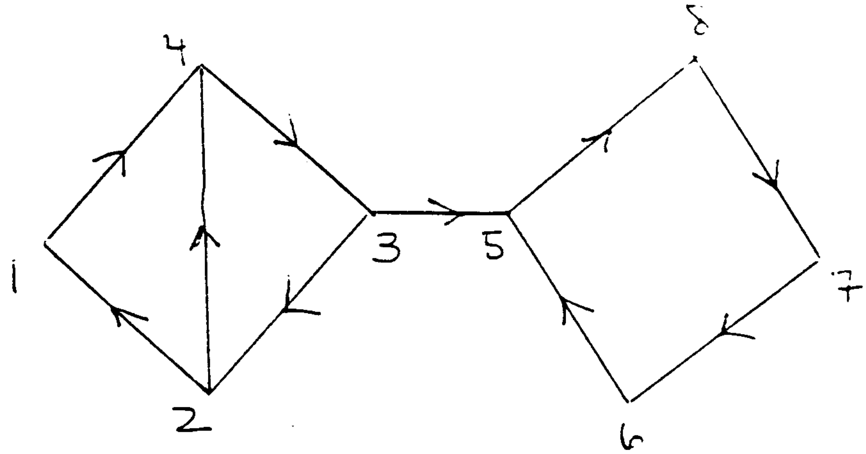
A **directed graph** or **digraph G** consists of a set **V** of **vertices** $\{1, 2, \ldots, n\}$ and a set **E** of edges: pairs of vertices (i, j). The edge (i, j) is considered to be directed from j to i. We assume i) there are no edges from a point to itself; that is no (i, i) is an edge. ii) there is at most one edge between two vertices: that is (i, j) is 0 or 1.

Let m be the number of edges. $(n, m)=$ (number of vertices, number of edges) is the **size** of the digraph.

We can draw digraphs by representing each vertex by a point, and each edge by a directed line (line-with arrowhead). The edge (i, j) is thus represented by an arrow from j to i.

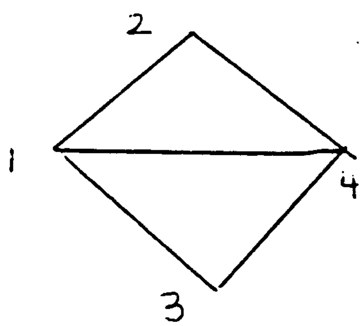


D 1

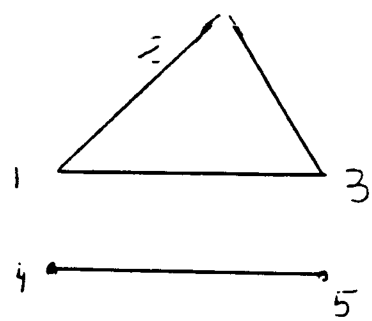


D2

For some pairs of vertices, there may be an arrow in both directions; that is, both $(i, j$ and (j, i) are edges. If all edges are two way, we get a **graph**, and need not draw the arrowheads.



G 1



G 2

The **incidence matrix** of a digraph is the $n \times n$ matrix M of 1s and 0's where we put a 1 in the (i, j) spot if there is an arrow from j to i; otherwise the entry is zero. That is: $m_{i,j} = 1$ if (i, j) is an edge, $m_{i,j} = 0$ if (i, j) is not an edge.

A **path** in a graph is a sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$. Note that the first term of each edge is the same as the second term of its predecessor. We call k the **length** of the path. A path is called **closed** if its end point is its beginning point: $i_k = i_1$.

The (i, j) entry of M^k (denoted $m^k_{i,j}$) is the number of paths of length k from j to i. For a graph this is the same as the number of paths of length k from i to j. The (i, j) entry of $M + M^2 + \cdots + M^k$ is the number paths of length at most k from i to j. Note that the (i, i) entry of M^k is the number of closed paths of length k starting and ending at i.

Check this with some of the digraphs above.

What is the number of distinct closed paths of length k in a digraph?

A digraph is called **connected** if there is a path from any one vertex to another. How can we tell if a digraph is connected?

[Notice that in the matrix M for D2, for all k, the (i, j) entry is zero if $i \leq 4$ and $j \geq 5$.]

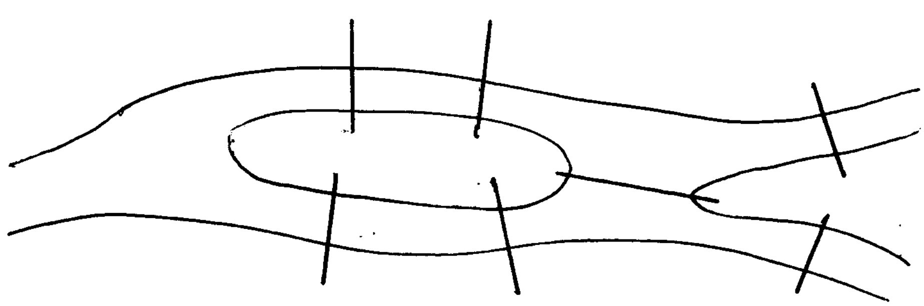
How do we find the length of the shortest path from one vertex, j, to another, i?

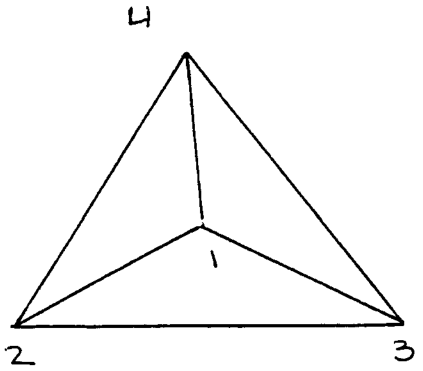
How do we find this shortest path? [Try G3 or G4.]

The Bridges of Königsberg. Is it possible to take a walk in Königsberg so as to cross each bridge exactly once? (see the next page). Here all the bridges are two-way bridges.

A **circuit** in a graph is a path which uses each edge at most once. A **complete circuit** is a path which uses each edge once and only once. It need not start and end at the same place; if it does, we'll call it a **closed complete circuit**.

When is a connected graph a complete circuit? This is the general Königsberg problem.





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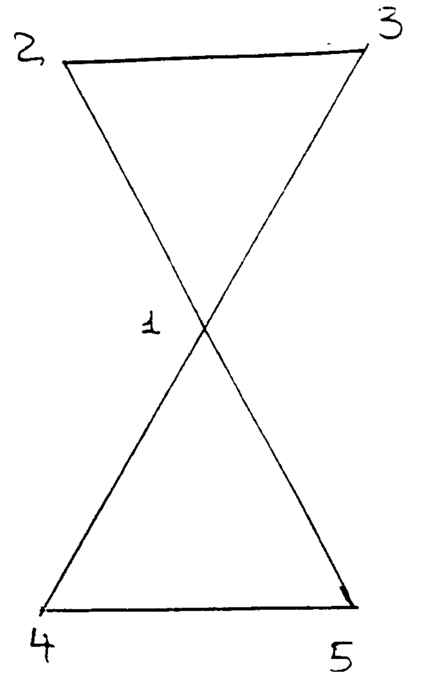
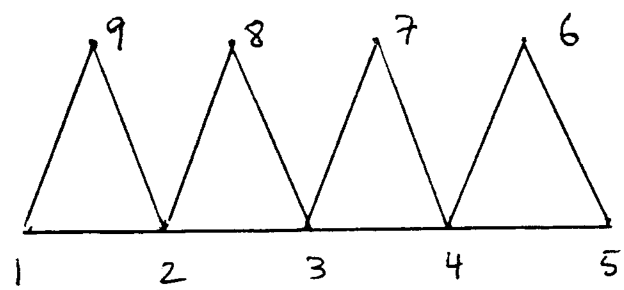


G 4



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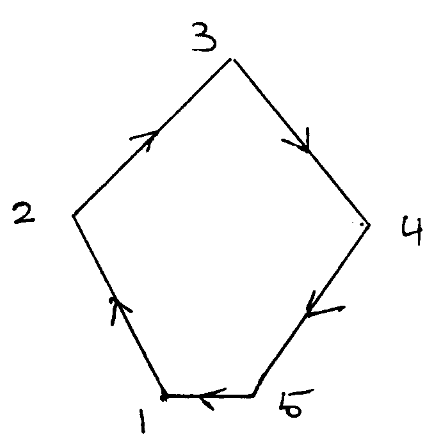
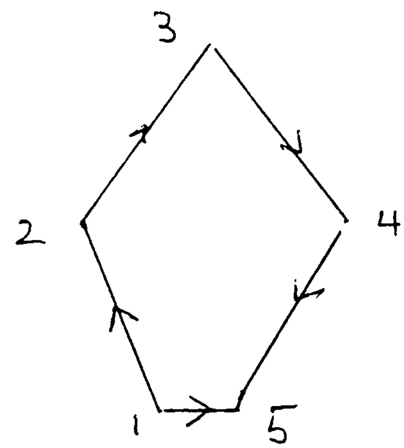


PG



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