Introductory Problems

Today we will solve problems that involve counting and probability. Below are problems which introduce some of the concepts we will discuss.

1. At one of George Washington's parties, each man shook hands with everyone except his spouse, and no handshakes took place between women. If 13 married couples attended, how many handshakes were there among these 26 people?

2. Find the number of positive integers not exceeding 1000 that are neither the square nor the cube of a positive integer.

3. How many ordered, nonnegative integer triples (x, y, z) satisfy the equation x + y + z = 11?

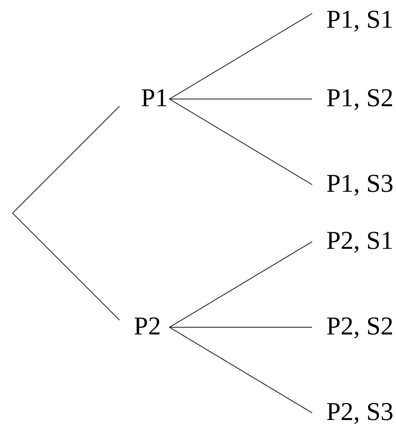
4. A circular table has exactly 60 chairs around it. There are N people seated around this table in such a way that the next person to be seated must sit next to someone. What is smallest possible value of N?

5. A bag contains a number of marbles of which 78 are red, 24 are blue, and the rest are green. If the probability of selecting a green marble is 1/3, what is the probability of selecting a red marble?

Introduction to Counting

We talked about some basic counting last fall in "Trax, Trains, and Trolleys". We begin today with a review of some of the concepts from that section.

Product Rule — One basic principle of counting is the product rule. Suppose we want to count the number of ways to pick a shirt and pair of pants to wear. If we have 3 shirts and 2 pairs of paints (a typical graduate student wardrobe), the total number of ways to choose an outfit is $2 \cdot 3 = 6$. Why? This can be seen by making a tree diagram. At the first branch we choose a pair of pants, and at the second branch we choose a shirt. The number of outfits is the number of leaves at the end of the tree. In general if you have n ways to choose the first and m ways the choose the second, independent of the first choice, there are nm ways to choose a pair.



Factorial — Suppose you want to count the number of ways to arrange 5 people in a line. Using the multiplication principle there are 5 ways to pick the first person, 4 ways to pick the next, and so on. The number of arrangements is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. These types of products come up often enough that we have a special notation for them: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. The expression $5!$ is read "five factorial". In general

$$n! = n \cdot (n-1) \cdot (n-2) \ldots 2 \cdot 1,$$

and we define

$$n! = 0.$$

Suppose you now want to make a line of five people but you now have 12 people to choose from. The solution using the multiplication rule is $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$. This can also be written as $12!/(12-5)!$.

Combinations — Notice in the previous example that we were lining people up, and so different orderings of the same people were counted separately. What if we wished to count the number of ways to pick five people from a group of 12, but the order of the five chosen did not matter? The number $12!/(12-5)!$ includes all the ways to choose five people when order matters. Once a group of 5 is picked there are $5!$ ways to order them, so the number $12!/(12-5)!$ counts each group $5!$ times. Dividing by this gives the correct count

$$\frac{12!}{(12-5)!5!}.$$

There is notation for numbers of this form

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

This is read "n choose k", and it represents the number of way to choose (hence the name) k objects from a set of n objects where the order does not matter.

Binomial Coefficients — You may be familiar with combinations from the binomial theorem.

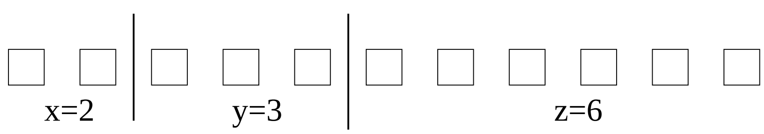
$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^n + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n,$$

or using summation notation

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k.$$

Can you explain the formula using counting?

What to count? — Often the challenge of a counting problem is deciding what to count! In problem 3 you are asked to count the nonnegative integer solutions of an equation. Think of solutions as dividing up 11 boxes into three groups. If we put the boxes on the table, we can form groups by inserting 2 dividers in any of 12 places (groups can be empty). The case for $(x, y, z) = (2, 3, 6)$ is shown below. Counting the dividers and boxes together, there are a total of 13 objects, and once we decide which 2 are dividers, a solution is determined. Therefore the solution is $\binom{13}{2}$.



Pigeonhole principle — "If $k+1$ or more objects are to be placed into k boxes, then there is at least one box containing two or more objects." This principle is quite easy to understand, but often the challenge is deciding what is a pigeon and what are the holes. Problem 4 illustrates the use of the principle. Because no one at the table is currently sitting next to anyone, there must be at least two empty seats adjacent to each person (one on the left and one on the right). So that each seated person occupies the maximal space, divide the table into groups of three seats with one person seated in the middle. There are 60/3=20 groups of three, and therefore $N = 20$. For this problem the people are pigeons and the sets of three chairs are the pigeonholes.

Inclusion Exclusion

Problem 2 can be solved using the principle of inclusion-exclusion. There are 1000 positive integers not exceeding 1000. So is the solution 1000-(# of perfect squares in [1,1000])-(# of perfect cubes in [1,1000])? Notice that numbers that are both perfect cubes and squares such as 1 and 64 were subtracted from 1000 twice. To get the correct count, these numbers must added back in (meaning they would be subtracted only once). The solution is

$$1000 - (\text{\# perfect sqares}) - (\text{\# perfect cubes}) + (\text{\# perfect sqares and cubes}).$$

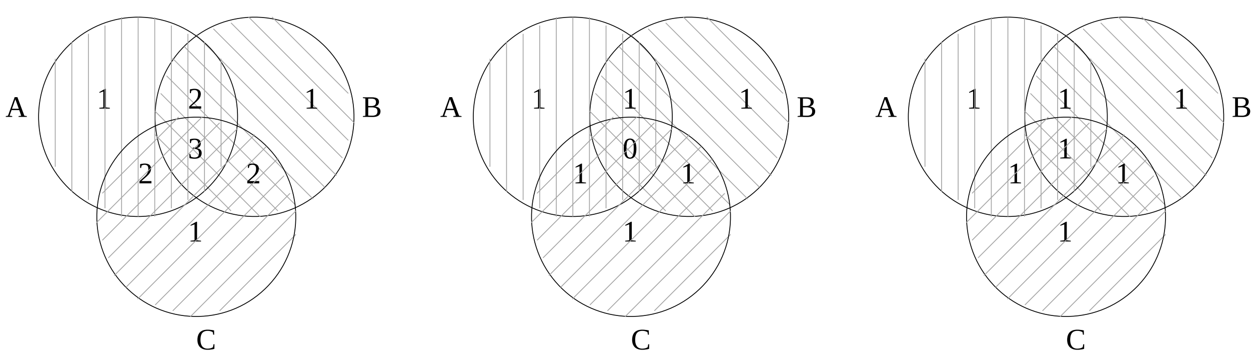
Before discussing the general principle in inclusion-exclusion, we introduce some notation. Let A and B be sets. The number of elements in A is denoted by $|A|$. The union of A and B is denoted by $A \cup B$ and is defined to be the set whose elements are in A or B. The intersection of A and B is denoted by $A \cap B$ and is defined to be the set whose elements are in both A and B. Following the same arguments used in the solution above,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Now consider three sets A, B, and C. Suppose we want to count the number of elements in all three sets ($|A \cap B \cap C|$). The number $|A| + |B| + |C|$ counts elements that are in two sets twice, and elements that are in three sets are counted three times. To correct for this over-count we subtract all elements in two sets, but now we subtracted the elements in three sets three times, so these must be added to the count. This is illustrated in the picture below. The correct count is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

This can be generalized to an arbitrary number of sets using induction. (Can you prove it?).



(a) Count of $|A| + |B| + |C|$

(b) Count of $|A|+|B|+|C| - |A \cap B| - |A \cap C| - |B \cap C|$

(c) Count of $|A|+|B|+|C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Probability

Frequency Probability — We consider here probabilities of events when the number of outcomes is discrete and finite. We start with the example of rolling a single die. In general the **sample space** is defined as the set of all outcomes. For the roll of the die, the sample space is $\{1, 2, 3, 4, 5, 6\}$. An **event** is any subset of the sample space. The probability of event E which is a subset of the finite sample space S of equally likely outcomes is

$$p(E) = \frac{\#\text{ of elements in } E}{\#\text{ of elements in } S}.$$

Suppose we wish to calculate the probability that rolling the die produces and even number. There are three ways the number on the die can be even, meaning the probability is $3/6 = 1/2$.

Independence — Two events A and B are independent if and only if the probability of $A \cap B$ (or A and B happen) is the product of the probabilities, $p(A)p(B)$. You can recognize independent events as ones for which the outcome of one has no effect on the outcome of the other. (Note the similarity with the product rule used for counting.) For example suppose we want to compute the probability of rolling a die twice and getting a 6 each time. The outcome of the first roll has no effect on the outcome of the second roll. Therefore the two rolls are independent. The probability of getting a 6 on each roll is $1/6$, and so the probability of getting successive 6's is $1/6 \cdot 1/6 = 1/36$.

Complementary Events — Denote the complement of event A by $\bar{A}$. The complement contains all the outcomes of the sample space not in A. Note that

$$p(A) + p(\bar{A}) = 1.$$

Sometimes computing the complement probability is easier. Once the complement probability is known, the identity above can be used to obtain the probability.

Unions of Events — Let E_1 and E_2 be two events in the sample space S. Then,

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

The above identity can be proved using inclusion–exclusion. Can you prove it?

Contest Problems

Moderate Problems

1. A mission to Mars will consist of 4 astronauts selected from 14 available. Exactly 5 of the 14 are trained in exobiology. If the mission requires at least 2 trained in exobiology, how many different crews can be selected?

   (a) 660 (b) 455 (c) 360 (d) 90 (d) 95

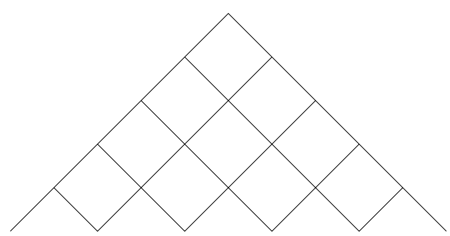
2. An $11 \times 11 \times 11$ wooden cube is formed by gluing together 11^3 unit cubes. What is the greatest number of unit cubes that can be seen from a single point?

   (a) 328 (b) 329 (c) 330 (d) 331 (e) 332

3. How many of the numbers, 100, 101,. . . ,999, have three different digits in increasing order or in decreasing order?

   (a) 120 (b) 168 (c) 204 (d) 216 (e) 240

4. Skiers at the top of the mountain have a variety of choices as they head down the trails. Assume that at each intersection, a skier is equally likely to go left or right. Find the percent of skiers who end up at C or D.



   (a) 62.5% (b) 50% (c) $33.\bar{3}\%$ (c) $57.\overline{142857}\%$ (e) not listed

5. Label one disk "1", two disks "2", three disks "3", . . . , fifty disks "50". Put these $1 + 2 + 3 + \cdots + 50 = 1275$ labeled disks in a box. Disks are then drawn at random without replacement. What is the minimum number of disks that must be drawn to guarantee drawing at least ten disks with the same label?

   (a) 10 (b) 51 (c) 415 (d) 451 (d) 501

6. The Clippers are playing the Warriors in a series of three basketball games. The Clippers' probability of winning any particular game is nonzero, independent of other games, and is 1.6 times as large as its probability of winning the series. What is the probability of the Clippers winning the series?

   (a) $\frac{5}{32}$ (b) $\frac{10}{32}$ (c) $\frac{15}{32}$ (d) $\frac{25}{32}$ (e) None of these

7. Suppose you are on a game show. At one point you are presented with three doors; behind only one of them is a valuable prize. You are asked to choose a door, and you choose door #1. The host opens one of the doors that you did not select and reveals that there is no prize. Then you are asked if you would like to take what is behind door #1 or take what is behind the other door. Find the probability of winning the valuable prize if you stick with your original guess of door #1.

Challenging Problems

8. First a is chosen from the set $\{1,2,3,\ldots,99,100\}$, and then b is chosen from the same set. The probability that the integer $3^a + 7^b$ has units digit 8 is

   (a) $\frac{1}{16}$ (b) $\frac{1}{8}$ (c) $\frac{3}{16}$ (d) $\frac{1}{5}$ (e) $\frac{1}{4}$

9. Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta, and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professors Alpha, Beta, and Gamma choose their chairs?

   (a) 12 (b) 36 (c) 60 (d) 84 (e) 630

10. A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?

    (a) $8!$ (b) $2 \cdot 8!$ (c) $(8!)^2$ (d) $\frac{16!}{2^8}$ (e) $16!$

11. A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

    (a) $\frac{3}{10}$ (b) $\frac{2}{5}$ (c) $\frac{1}{2}$ (d) $\frac{3}{5}$ (e) $\frac{7}{10}$

Very Challenging Problems

12. Find $\sum_{k=0}^{49}(-1)^k\binom{99}{2k}$.

    (a) -2^{50} (b) -2^{49} (c) 0 (d) 2^{49} (e) 2^{50}

13. For any set S, let $|S|$ denote the number of elements in S, and let $n(S)$ denote the number of subsets of S, including the empty set and the set itself. If A, B, and C are sets for which

$$n(A) + n(B) + n(C) = n(A \cup B \cup C) \text{ and } |A| = |B| = 100,$$

then what is the minimum possible value of $|A \cap B \cap C|$?

(a) 96 (b) 97 (c) 98 (d) 99 (e) 100

14. Suppose that 6 boys and 9 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row GBGGBBGBBGGGBGG we have $S = 8$. The average value of S (if all possible orders of these 15 people are considered) is closest to

(a) 6 (b) 7 (c) 8 (d) 9 (e) 10

15. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

Solutions

Note: There are often many ways to arrive at the same solution. Sometimes multiple solutions are given; however, these solutions do not represent all methods for solving the problems. You may find more clever ways to solve them.

1. Break this into three cases: teams with 2,3, and 4 exobiologists. For each case, choose the team of exobilogists; then choose the rest of the team from the 9 not trained in exobiology.

$$\begin{aligned}\binom{5}{2}\binom{9}{2}+\binom{5}{3}\binom{9}{1}+\binom{5}{4}\binom{9}{0} &= \frac{5!}{3!2!}\frac{9!}{7!2!}+\frac{5!}{2!3!}\frac{9!}{8!1!}+\frac{5!}{1!4!}\frac{9!}{9!0!} \\ &= 10\cdot 36+10\cdot 9+5\cdot 1 \\ &= 455\end{aligned}$$

2. At most three of the large cube's six faces can be seen at once. The cubes on these three faces may be counted using the inclusion-exclusion principle as follows, (# unit cubes on each face) − (# unit cubes on two faces) + (# unit cubes on all three faces). This sum is $3\cdot 11^2-3\cdot 11+1=331$.

3. For every 3 distinct digits selected from $\{1,2,\ldots,9\}$ there is exactly one way to arrange them into a number with increasing digits, and every number with increasing digits corresponds to one of these selections. Similarly, the numbers with decreasing digits correspond to the subsets with 3 elements of the set of all 10 digits. Hence, the answer is

$$\begin{aligned}\binom{9}{3}+\binom{10}{3} &= \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}+\frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} \\ &= \frac{9\cdot 8}{3\cdot 2}(7+10) \\ &= 204\end{aligned}$$

4. At each turn there is the choice of going left or right. The total number of paths down the mountain is 2^5. To end up at point C the skier must make 2 left turns and 3 right turns, and to end up at point D the skier must make 3 left turns and 2 right turns. To count the number of paths, choose from the five turns when to make the left turns. The proportion of skiers who end up at C or D is

$$\frac{\binom{5}{2}+\binom{5}{3}}{2^5}=\frac{5}{8}=62.5\%.$$

5. We can draw all of the disks with labels "1" through "9", because there are not 10 of any of these disks. There are $1+2+\cdots+9=45$ of these types of disks. From the rest of the disks 41 disks, we can draw 9 of each type for a total of $41\cdot 9=369$. The total number of disks drawn is $45+369=414$. If another disk is drawn, its label must match one of the 369 disks already drawn.

6. Let p denote the probability that the Clippers win a given game, and so the probability they lose a given game is $1 - p$. There are three ways the clippers can win the series. These possibilities are listed below along with their probabilities.

series	probability
CWC	$p(1-p)p$
WCC	$(1-p)p^2$
CC	p^2

Summing these probabilities gives the probability of winning the series, which must be $p/1.6$. We must solve the equation

$$2p^2(1-p) + p^2 = p/1.6.$$

This equation has the three solutions $p = 0$, $1/4$, and $5/4$. It is given that the probability is not zero, and $5/4$ is not a probability. Therefore $p = 1/4$. The probability of winning the series is $p/1.6 = 5/32$.

7. Let's label the outcomes using the notation "P1H2" to denote the case of "the prize is behind door 1 and host opens door 2". First suppose the prize is behind door 1. The host has a choice of whether to open door 2 or door 3. Suppose that the probability is $1/2$ for each. Therefore the probability of P1H2 is $1/3 \cdot 1/2 = 1/6$, and similarly for P1H3. Now suppose that the prize is behind door 2. Because you have chosen door 1, the host is obligated (probability 1) to open door three, making the probability of P2H3 $1/3 \cdot 1 = 1/3$. By the same argument the probability of P3H2 is also $1/3$. These are all the possible outcomes. The probability you will get the prize if you stick with door 1 is $1/6 + 1/6 = 1/3$.

8. Observe the repeating pattern of the units digits of consecutive integral powers of 3 and 7:

$$\begin{array}{rr} 3^1 = 3 & 7^1 = 7 \\ 3^2 = 9 & 7^1 = 49 \\ 3^3 = 27 & 7^1 = 343 \\ 3^4 = 81 & 7^1 = 2401 \\ 3^5 = 243 & 7^1 = 16807 \end{array}$$

Note that 25 of the given values for a yield a units digit in 3^a of 3, 25 yield 9, 25 yield 7, and 25 yield 1. Similarly for 7^b, each of the possible units digits can be obtained in 25 ways. There are 16 possible pairs of units digits which are all equally likely. The three pairs which give units digit 8 for $3^a + 7^b$ are $(1, 7)$, $(9, 9)$, and $(7, 1)$. The probability is therefore $3/16$.

9. Imagine the six students standing in a row before they are seated. There are 5 spaces between them, each of which may be occupied by at most one of the three professors. The arrangement may be counted by assigning a different position to each professor. There are $5!/3! = 5 \cdot 4 \cdot 3 = 60$ ways for the professors to select their places.

10. Ignoring that a sock must be on a foot before a shoe, there are 16! ways the spider can put on its shoes and socks. Consider just one of the spider's legs. In exactly half of the 16! cases the sock was put on that leg before the shoe. In the other half of the cases, the shoe was put on before the sock. Therefore there are $16!/2$ cases in which the sock went on before the shoe for this leg. Repeating this argument for each of the other 7 legs gives that there are $16!/2^8$ ways for the spider to put on its shoes and socks with the shoes and socks put on in the proper order for each leg.

11. Think of continuing the drawing until all five chips are removed from the box. There are $\binom{5}{2} = 10$ possible orderings (choose when the whites were drawn), which are all equally likely. There are $\binom{4}{2} = 6$ orderings where the last chip drawn was red. These are exactly the cases for which all the white chips were drawn before all the red chips. Therefore the probability is $6/10 = 3/5$.

12. By the binomial theorem,

$$(1+i)^{99} = \binom{99}{0} + \binom{99}{1} i + \binom{99}{2} i^2 + \cdots + \binom{99}{99} i^{99}.$$

Note the real part of this series is

$$\binom{99}{0} - \binom{99}{2} + \binom{99}{4} - \binom{99}{6} + \cdots \binom{99}{98},$$

which is the sum we are asked to find. If you are familiar with complex exponentials, compute the sum by

$$\begin{aligned} (1+i)^{99} &= \left(\sqrt{2}\, \mathrm{e}^{\mathrm{i}\,\pi/4}\right)^{99} \\ &= 2^{99/2}\, \mathrm{e}^{\mathrm{i}\,99\pi/4} \\ &= 2^{99/2} \left(\cos\left(99\pi/4\right) + i\, \sin\left(99\pi/4\right)\right) \\ &= 2^{99/2} \left(\cos\left(3\pi/4\right) + i\, \sin\left(3\pi/4\right)\right) \\ &= 2^{99/2} \left(-1/\sqrt{2} + i\, 1/\sqrt{2}\right) \end{aligned}$$

The real part is $-2^{99/2-1/2} = -2^{-49}$. If you are not familiar with complex exponentials, you may be familiar with Demoivre's Theorem. If so, ignore the first line of the computation above. You may also try writing down the first few powers of $(1+i)$ and to find the pattern.

13. If a set has k elements, then it has 2^k subsets. We are given

$$\begin{aligned} 2^{100} + 2^{100} + 2^{|C|} &= 2^{|A \cup B \cup C|} \\ 2^{101} + 2^{|C|} &= 2^{|A \cup B \cup C|} \\ 1 + 2^{|C|-101} &= 2^{|A \cup B \cup C|-101}. \end{aligned}$$

The left side, $1 + 2^{|C|-101}$, is larger than 1, and so the right side must also be larger than 1. Therefore $|A \cup B \cup C| - 101 > 0$, and it is an integer. Thus the left side, $1 + 2^{|C|-101}$, must be an integral power of 2. The only integral power of 2 of the form $1 + 2^m$ is $1 + 2^0 = 2^1$. Hence

$$|C| = 101 \quad \text{and} \quad |A \cup B \cup C| = 102.$$

Using the inclusion–exclusion principle,

$$\begin{aligned} |A \cap B \cap C| &= |A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| \\ &= -199 + |A \cap B| + |A \cap C| + |B \cap C|. \end{aligned}$$

Using the 2 set inclusion–exclusion principle,

$$\begin{aligned} |A \cap B \cap C| &= (|A| + |B| - |A \cup B|) + (|A| + |C| - |A \cup C|) \\ &\quad + (|B| + |C| - |B \cup C|) - 199 \\ &= 2(|A| + |B| + |C|) - 199 \\ &\quad - (|A \cup B| + |A \cup C| + |B \cup C|) \\ &= 403 - (|A \cup B| + |A \cup C| + |B \cup C|). \end{aligned}$$

Because each union of two sets in contained in the the union of all three sets, we have

$$|A \cup B|, |A \cup C|, |B \cup C| \leq 102.$$

This gives a lower bound

$$\begin{aligned} |A \cap B \cap C| &= 403 - (|A \cup B| + |A \cup C| + |B \cup C|) \\ &\geq 403 - 3 \cdot 102 = 97. \end{aligned}$$

To show this lower bound can be obtained, consider the example

$$A = \{1, 2, \ldots, 100\}, \quad B = \{3, 4, \ldots, 102\}, \quad C = \{1, 2, \ldots 102\}.$$

14. Begin by counting the number of ways a boy and girl can be standing in the first and second positions in line. Call this number N_1. There are $6 \cdot 9$ pairs of a boy with a girl and 2 ways to order them, and there are 13! ways to order the remaining people. Therefore, $N_1 = 2 \cdot 6 \cdot 9 \cdot 13!$. In general, let N_j be the number of ways a boy and girl can be next to each other in the j^{th} and $(j+1)^{st}$ positions. For each $j = 1, \ldots, 14$,

$N_j = 2 \cdot 6 \cdot 9 \cdot 13!$. The average is the sum of the N_j divided by the total number of orderings, 15!. This is

$$\begin{aligned} \frac{N_1 + N_2 + \cdots + N_{14}}{15!} &= \frac{14 \cdot (2 \cdot 6 \cdot 9 \cdot 13!)}{15!} \\ &= \frac{2 \cdot 6 \cdot 9}{15} \\ &= \frac{36}{5} \\ &= 7.2 \end{aligned}$$

This is closest to 7.

15. Let's compute the probability that Shanille continues the pattern of a made free throw followed by a missed free throw until she has shot 100 shots total. This probability is

$$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{2}{5} \cdot \frac{3}{6} \cdot \frac{3}{7} \cdots \frac{49}{98} \cdot \frac{49}{99} = \frac{(49!)^2}{99!}.$$

This is one possible way in which she makes exactly 50 out of 100 of her first shots. For any other sequence of 100 shots, the denominators for each shot's probability will be identical to those above, since they are simply the number of previous attempts. For the j^{th} made free throw, the numerator of its probability is $j-1$. Similarly for the j^{th} miss. Since there are exactly 49 made shots and 49 missed shots after the first two shots, each integer from 1 to 49 appears as the numerator of a probability exactly twice. The ordering of the numerators will be different for each sequence, but the product of all the numerators will not change. Therefore any sequence with exactly 50 made shots of the first 100, with the first two as given, has probability of occurring $\frac{(49!)^2}{99!}$. The number of such sequences is $\binom{98}{49}$. Therefore the probability is

$$\binom{98}{49} \frac{(49!)^2}{99!} = \frac{98!}{49!49!} \cdot \frac{(49!)^2}{99!} = \frac{98!}{99!} = \frac{1}{99}$$