SHARE THE WEALTH[1]

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Mathematics is like checkers in being suitable for the young, not too difficult, amusing, and without peril to the state

— Plato (circa 400 BC)

Consider the following game which was the subject of a problem from the 1962 Beijing Mathematical Olympiad. A number of students sit in a circle. Their teacher gives each of them some candy. Each student receives an even number of pieces. Then the teacher rings a bell, and each student passes half of his pile of candy to the student on his right. Each student who now has an odd number of pieces of candy is given an extra piece so that their pile again consists of an even number. Then the teacher rings a bell and the process begins again. The game stops when all students have the same amount of candy[2].

One can immediately ask if there is any reason to expect the game to end at all. If it does end, one can ask how many moves (that is, bell rings) it takes to reach the end of the game. (This will clearly depend on the initial distribution of candy.) And so on.

To get a feel for the game, lets consider a particular example where there are four students (Alice, Bert, Candice, and David) seated in a circle so that Bert is at Alice's right, Candice is at Bert's right, David is at Candice's right, and finally Alice is at David's right. Suppose that initially Alice has 8 pieces, Bert has 10, Candice has 2, and David has 0. The bell ring. Of Alice's 8, she gives away 4 to Bert, takes 0 from David, and is thus left with $8-4+0=4$. Meanwhile of Bert's 10, he gives 5 to Candice, receives 4 from Alice, and is left with $10-5+4=9$, which is odd. So he gets one additional one from the teacher to arrive at 10 again. Similarly Candice ends with $2-1+5=6$, and David ends with $0-0+1(+1)=2$.

We can iterate the process and keep track of our results in the following table.

A	B	C	D
8	10	2	0
$4=8-4+0$	$10=10-5+4(+1)$	$6=2-1+5$	$2=0-0+1(+1)$
$4=4-2+1(+1)$	$8=10-5+2(+1)$	$8=6-3+5$	$4=2-1+3$
$4=4-2+2$	$6=8-4+2$	$8=8-4+4$	$6=4-2+4$
$6=4-2+3(+1)$	$6=6-3+2(+1)$	$8=8-4+3(+1)$	$8=6-3+4+1$
$8=6-3+4(+1)$	$6=6-3+3$	$8=8-4+3(+1)$	$8=8-4+4$
$8=6-3+4(+1)$	$8=6-3+4+1$	$8=8-4+3(+1)$	$8=8-4+4$
8	8	8	8

Thus the above game really did end in a finite number of steps. (It took only 6 rings of the bell.) Did we get lucky? What if we had started with a different initial distribution of candy? How about playing the game with 5 players? 10 players? 100?! Here are some exercises to get warmed up.

[1]A good reference for this material is the article "Candy Sharing," by G. Iba and J. Tanton, *American Math. Monthly*, volume 110 (2003), no. 1, pp. 25–35.

[2]Note the formal similarity of this game with the one we considered last semester under the heading of "Diffy Boxes." See the notes available at `http://www.math.utah.edu/mathcircle/notes/diffybox.pdf`. It would be a wonderful research project to relate these two games directly.

EXERCISES I: FIRST EXAMPLES

1. Repeat the above analysis for the four-player game but instead assume that Alice begins with 20 pieces, Bert has 0, Candice has 4, and David also has 4.

2. Repeat problem 1 but instead assume Alice has 30 pieces initially, while all the others have 0.

3. Now play a five player version of the game with the following initial candy totals: Alice start with 0, Bert with 12, Candice with 8, David with 20, and Ernie with 6.

4. Suppose that there are only two players, Alice and Bert. What is the maximum number of moves needed before the game ends (given any initial configuration of candy)?

5. Consider the three person version of the game. Suppose Alice starts with 1000 pieces, and Bert and Candice start with 0. How many moves are requires before the game ends?

6. What is the maximum number of moves needed to finish the three-person game (given any initial configuration of candy)?

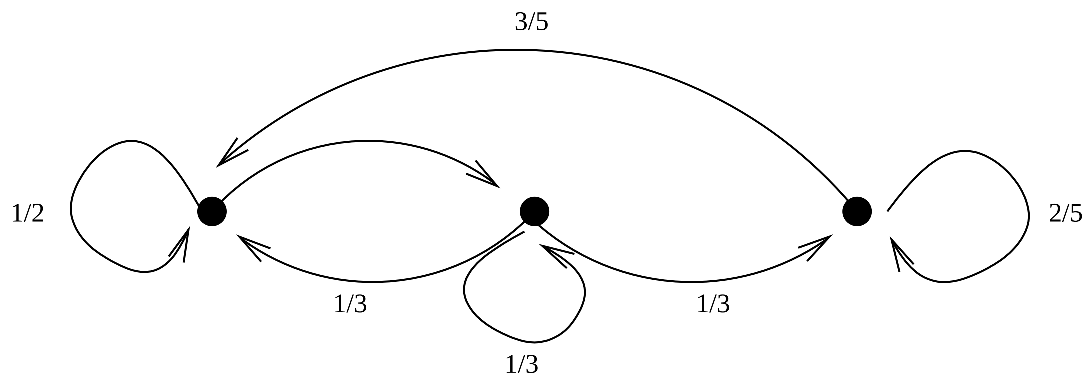


FIGURE 1. A graphical representation of a generalized three player game.

These examples certainly suggest that the original game always ends with a finite number of moves. We are going to give a proof of this (and discuss more general results), but we first begin by studying a more general class of games. Here is the setting. Suppose we have n players $1, 2, \ldots, n$. Each player i is assigned a number p_i, and a list of n numbers

$$a_{i,1}, a_{i,2}, \ldots, a_{i,n}$$

between 0 and p_i so that

$$a_{i.1} + a_{i,2} + \cdots + a_{i,n} = p_i,$$

for each player i. The game starts with each player receiving a pile of candy so that the ith player's pile contains a number of pieces that is divisible by p_i. The teacher rings the bell. For each j, player i multiplies the number of pieces in his pile by $a_{i,j}/p_i$ and passes that amount to player j. The teacher then comes around and adds additional pieces to each players pile so that each player i has an amount divisible by p_i. Then the bell rings again and the process begins anew.

For example, in our the game initially described above, there are n players, $p_i = 2$ for each of them and for $1 \le i \le n-1$

$$a_{i,j} = \begin{cases} 1/2 & \text{if } j = i \text{ or } i + i \\ 0 & \text{otherwise} \end{cases}$$

while

$$a_{n,j} = \begin{cases} 1/2 & \text{if } j = n \text{ or } 1 \\ 0 & \text{otherwise.} \end{cases}$$

We can represent the general game graphically. As an example, consider the three person game where

$$p_1 = 2, p_2 = 3 \text{ and } p_3 = 5$$

and

$$a_{1,1} = a_{1,2} = 1$$
$$a_{2,1} = a_{2,2} = a_{2,3} = 1$$
$$a_{3,1} = 3, \qquad a_{3,3} = 2,$$

while all other $a_{i,j}$'s are 0. We may represent this graphically as in Figure 1.

If Player 1 start with 6 pieces, Player 2 starts with 9, and Player 3 start with 10, we can play the game as follows.

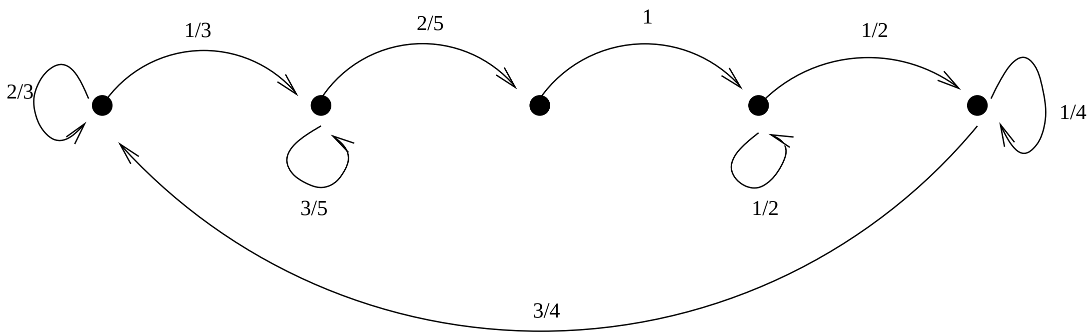
Player 1	Player 2	Player 3
6	9	10
$12 = 6 - 3 + 3 + 6$	$6 = 9 - 6 + 3$	$10 = 10 - 6 + 3(+3)$
$14 = 12 - 6 + 2 + 6$	$9 = 6 - 4 + 6(+1)$	$10 = 10 - 6 + 2(+4)$
$16 = 14 - 7 + 6 + 3$	$12 = 9 - 6 + 7(+2)$	$10 = 10 - 6 + 3(+3)$
$18 = 16 - 8 + 6 + 4$	$12 = 12 - 8 + 8$	$10 = 10 - 6 + 4(+2)$
$20 = 18 - 9 + 6 + 4(+1)$	$15 = 12 - 8 + 9(+2)$	$10 = 10 - 6 + 4(+2)$
$22 = 20 - 10 + 6 + 5(+1)$	$15 = 15 - 10 + 10$	$10 - 6 + 5(+1)$
$22 = 22 - 11 + 6 + 5$	$18 = 15 - 10 + 11(+2)$	$10 = 10 - 6 + 5(+1)$
$24 = 22 - 11 + 6 + 6(+1)$	$18 = 18 - 12 + 11(+1)$	$10 = 10 - 6 + 6$
24	18	10

The next set of exercises deals with this more general game.

EXERCISES II: THE MORE GENERAL SETTING

1. Repeat the above example, but instead assume that Player 1 begins with 50 pieces, Player 2 with 0, and Player 3 with 100.

2. Consider the following five player game where the "rounding up" values are $p_1 = 3, p_2 = 5, p_3 = 1, p_4 = 2$ and $p_5 = 4$.



Determine the number of steps to finish the game if Player 1 begins with 12 pieces, Player 2 with 10, Player 3 with 3, Player 4 with 4, and Player 5 with 8.

3. Repeat the previous problem with the following initial distribution: Player 1 begins with 3 pieces, Player 2 with 30, Player 3 with 8, Player 4 with 12, and Player 5 with 8.

4. Devise a three player game (involving all three players) of the above type which *never* finishes? Can you find a general construction of such games?

Let's return to the original game described at the very beginning. The Beijing Olympiad problem asks for a proof that the game finishes in a finite number of steps no matter what the initial distribution of candy is. Here is a very clever way to see that. Let M denote the number of pieces in the possession of any one student at the outset. When the bell rings, a student with k pieces passes $k/2$ pieces to his right. The most he can receive from the person on his left is $M/2$ (since M was assumed maximal). Thus the most he can end up with is M (if he started with $k = M$ pieces and received $M/2$). Since M was assumed to be even, and he doesn't receive an additional piece from the teacher if he winds up with M. On the other hand, if he winds up with less than M, then the most he can obtain (even after possibly receiving an additional piece form the teacher) is indeed M. Thus we conclude that the game is bounded in the sense that *no student ever attains more than M pieces of candy.*

Now let's investigate the lower bound. Let m denote the smallest number of pieces in the possession of any one student at the outset. If a student begins with k pieces, when the bell rings he is left with an amount greater than or equal to $k/2 + m/2$ (since m was assumed to be minimal). The only way his pile could fail to grow after the bell rings is if he started with m pieces and the person on his left also started with m pieces. Thus if there are l people sitting in a row, each with m pieces, after the bell rings the last $l - 1$ will remain with m pieces while the first will have more than m pieces (unless everyone in the circle has m pieces, in which case the game would have already ended). Thus *the number of students with the smallest piles of candy decreases after each ringing of the bell.*

Since we have seen that there is an upper bound on the size of each pile of candy, the game must eventually stabilize. If it stabilizes, it is easy to see that each student must indeed have the same amount of candy. This solves the original problem and suggests another one:

Problem. *Describe a class of generalized games for which the above argument applies to show the game stabilizes.*

In fact, the above argument is not enough to analyze the most general case, and some more subtle reasoning is needed. I will simply describe the result here. Suppose one is given a generalized game represented graphically as above. Suppose that the game satisfies two requirements. First suppose that given any two players i and j, there is a directed path in the graph from i to j. Second suppose that there is at least one self-arrow; that is, suppose that there exists some player who does not give all of his candy away on each turn. Then one has the following theorem: the game stabilizes in a finite number of steps.

Notice that the original game, and indeed all those considered above (besides the ones you constructed in Problem 4 of the second set of exercises), satisfy the two hypotheses of the previous paragraph, and hence the general theorem implies that they eventually stabilize.