Recursion Relations and Qualitative Behavior of Sequences

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1 Introduction

Sequences play an important role in applied mathematics. They are often used as modeling tools for certain kinds of populations. When sequences are used in this manner, the general behavior of the sequence becomes an important quality that we would like to characterize. My goal today is to show you how to make these behavior assessments. We will start by reviewing some basic sequence information, including closed and recursion relation representations of sequences. Next, I will introduce you to a graphical method called cobwebbing that aids in assessing the behavior of sequences defined by some recursion relation. If time allows, we will study graphs of selected sequences to gain a basic understanding of how sequences can behave and what this behavior may depend on.

2 Representations of Sequences

A **sequence** is a particular ordering of objects that is indexed by the natural numbers ($\mathbb{N}$) or the non-negative integers ($\mathbb{Z}^+ + \{0\}$). Sequences can be composed of real numbers, functions, or sets and are often expressed as strings of these objects. Sometimes sequences can be expressed more formally by constructing a special function and/or recursion relation. Let's look at a couple of examples to futher illustrate these ideas.

Example 1. The sequence $1, 3, 5, 7, 9, \ldots$ is the sequence of odd integers beginning with 1. If we let n be any natural number and $a(n) = a_n$ be the n^{th} term of the sequence, then $a(n) = a_n = 2n - 1$ is a function that explicitly defines this sequence. This represenation is often called the **closed form** representation. We could also use the recursion relation $a_{n+1} = a_n + 2$ coupled with $a_1 = 1$ to define this sequence formally. Below is a side by side check that each representation independently recreates the sequence of odd integers as I have indicated they do.

Sequence Term a_n	Closed Form $a_n = 2n - 1$	Recursion Relation $a_{n+1} = a_n + 2, a_1 = 1$
$a_1 = 1$	$2(1) - 1 = 1$	1
$a_2 = 3$	$2(2) - 1 = 3$	$1 + 2 = 3$
$a_3 = 5$	$2(3) - 1 = 5$	$3 + 2 = 5$
$a_4 = 7$	$2(4) - 1 = 7$	$5 + 2 = 7$
$\vdots$	$\vdots$	$\vdots$

Example 2. The sequence $x, x^2, x^3, x^4, \ldots$ is the sequence of power functions beginning with x. This sequence differs from the first example in that it is a sequence composed of functions rather than real numbers. If we let n be any natural number and $a(n) = a_n$ be the n^{th} term of the sequence, then the closed form representation of this sequence is $a_n = x^n$. The recursion relation $a_{n+1} = xa_n$ coupled with $a_1 = x$ also defines this sequence. Again, I provide a side by side check to illustrate that each representation independently recreates the sequence of power functions.

Sequence Term a_n	Function (Closed Form) $a_n = x^n$	Recursion Relation $a_{n+1} = xa_n, a_1 = x$
$a_1 = x$	x^1	x
$a_2 = x^2$	x^2	$x(x) = x^2$
$a_3 = x^3$	x^3	$x(x^2) = x^3$
$a_4 = x^4$	x^4	$x(x^3) = x^4$
$\vdots$	$\vdots$	$\vdots$

Example 3. Let n be any natural number and $a(n) = a_n$ be the n^{th} term of a sequence. The sequence $2, 4, 8, 16, 32, \ldots$ has the representations $a_n = 2^n$ and $a_{n+1} = 2a_n$ when coupled with $a_1 = 2$.

Sequence Term a_n	Closed Form $a_n = 2^n$	Recursion Relation $a_{n+1} = 2a_n, a_1 = 2$
$a_1 = 2$	$2^1 = 2$	2
$a_2 = 4$	$2^2 = 4$	$2(2) = 4$
$a_3 = 8$	$2^3 = 8$	$2(4) = 8$
$a_4 = 16$	$2^4 = 16$	$2(8) = 16$
$\vdots$	$\vdots$	$\vdots$

Recursion relations are widely regarded as less satisfying representations of sequences. Why do you think this is so? Take a closer look at the recursion relation we constructed in Example 3. What if you were asked to compute the 10^{th} term of this sequence given only the recursion relation $a_{n+1} = 2a_n$ coupled with $a_1 = 2$ (and not the alternative closed form representation)? What information would you need to compute this term?

$$a_{10} = 2a_9$$

You would need the 9^{th} term, and for the 9^{th} term you would need the 8^{th}, and so on. You must be given the terms a_1 through a_9 before you could determine the value for a_{10}. This process could be tedious. However, if you were given the closed form representation, $a_n = 2^n$, computing the 10^{th} term would be a piece of cake since all you would have to do is plug $n = 10$ into a_n. Consequently, when you want to compute high or many terms of the sequence, choosing the recursion relation over the closed form is not a good decision!

If you would like more information on sequences or their representations, I encourage you to visit the website `http://www.research.att.com/~njas/sequences/`. This website allows you to input a particular integer sequence and it outputs known information about this sequence, including known representations and programming notes for this sequence.

PROBLEM SET 1

Directions: Attempt to find a function and a recursion relation that can be used to recreate each real number sequence.

1. $0, 1, 2, 3, 4, 5, \ldots$

2. $17, 17, 17, 17, 17, 17, \ldots$

3. $5, 3, \frac{9}{5}, \frac{27}{25}, \frac{81}{125}, \ldots$

4. $3, 9, 27, 81, 243, \ldots$

5. $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \frac{3}{16}, \ldots$

6. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

7. $1, 0, -1, 0, 1, 0, -1, 0, \ldots$

8. $-1, 7, 47, 223, 959, 3967, \ldots$

9. $1, 1, 2, 3, 5, 8, 13, \ldots$

3 Updating Functions, Initial Conditions, and Sequence Behavior

Recursion relations are often used in applied mathematics and in the sciences as representations of sequences. Here, the sequence a_n is usually taken to be the number of individuals within a population at some time step n and a recursion relation is constructed to model how this population is thought to change between successive time steps. It is this dynamical property that makes recursion relations appealing to modelers. These models show up in biology, anthropology, and physics. Whatever the application may be, the purpose of constructing a model is to somehow use it to predict future states of the population. The model may be able to predict population extinctions of massive expansions.

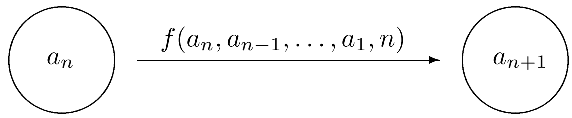
Do you see the problem yet? Even though recursion relations are great for modeling, they are terrible at providing explicit information about the sequence. At best, it would be difficult to determine the long-term trends of the population. There are a couple of options we can pursue. (1) We can compute the sequence explicitly by iterating the recursion relation until we feel we have obtained enough terms to make accurate predictions. This technique can be very time consuming and computationally expensive. Furthermore, if we are not fastidious, we may make a poor or even incorrect assessment. (2) We can use the recursion relation to find a closed form representation which we could in turn analyze. This technique can be mathematically taxing. (3) The last avenue is to figure out a way to use the recursion relation to tease out the long-term dynamics of the sequence without explicitly calculating the sequence or the closed form representation for the sequence. Remember that the recursion relation dictates how the sequence changes between any two time steps. Maybe there is a clever way to use this information to understand how the sequence behaves in general or just for large n.

Let's review recursion relations before we continue. Let n be any natural number and let a_n be the n^{th} term of some sequence. A **recursion relation** defines the process one must go through to get from any term of the sequence, say a_n, to the next one, a_{n+1}. This process may depend on any or all of the previous sequence terms $(a_1, a_2, \ldots, a_n)$ as well as the index variable n. In the language of mathematics, recursion relations take on the form

$$a_{n+1} = f(a_n, a_{n-1}, \ldots, a_1, n)$$

where f is a function that defines the before-mentioned process. The function f is sometimes called an

updating function as it uses known information about the sequence to produce new information about the sequence. In the context of population modeling, f serves as a description of the forces acting on a population between successive generations, like births and deaths.



Recursion relations alone cannot be used to represent a sequence. You must also provide some additional information. The recursion relation $a_{n+1} = 2a_n$ requires a_1 is specified while $a_{n+2} = a_n + a_{n+1}$ requires a_1 and a_2 are specified. An **initial condition** is any information that must be specified so that a recursion relation can be implemented.

3.1 Characterizing Long-Term Behavior

Today, we will learn a cool technique which reveals the long-term behavior of sequences with recursion relations of the form

$$a_{n+1} = f(a_n).$$

These forms are nice in that the updating function only depends on the current state. Discrete time mathematical models in the life sciences almost always take on this form.

Cobwebbing is a graphical method that can be used to quickly and almost effortlessly reveal the behavior of the recursion relation $a_{n+1} = f(a_n)$ coupled with any initial condition a_1. The general idea is as follows:

1. First, to find a_2 from a_1, we use the fact that a_2 is the result of applying updating function to a_1. If we were to consider a graph of the updating function, we could then easily identify a_2 with the y-coordinate associated with the point on the graph of the function directly above a_1. A similiar relation holds for all consecutive terms of the sequence.

2. Second, the axes of the above graph have special meaning. The horizontal axis represents the current term of the sequence while the vertical axis respresents the new or updated term. To proceed from a_2 to a_3, we need to somehow get a_2 from the new axis to the current axis while preserving its value. Once we have made this move, we can identify a_3 in the same manner we found a_2 from a_1. What

happens if we reflect current terms off the diagonal line $a_{n+1} = a_n$? We would be moving the point (a_1, a_2) horizontally until it intersects the diagonal line. This means we are now at (a_2, a_2). If we move vertically until we intersect the updating function, we are at point (a_2, a_3). This reflection effectively makes the a_2 transfer we wanted and it also aids in finding a_3.

Cool! Now we use this diagram to observe what happens to our sequence as we continue cobwebbing.

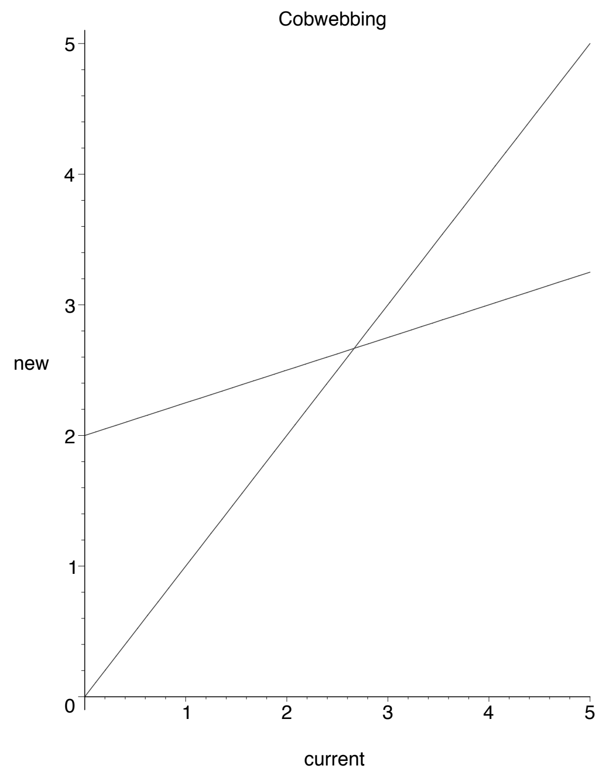


Figure 1: Graph of $a_{n+1} = f(a_n)$ and diagonal line $a_{n+1} = a_n$

Let's practice performing the technique and reading the resulting diagram for a particular updating function.

Example 4. Discuss the long-term behavior of the sequence defined by $a_{n+1} = 4a_n - 9$ with the initial condition $a_1 = 4$. What if we use the alternative initial conditions $a_1 = 2$ or $a_1 = 3$? How do these sequences behave?

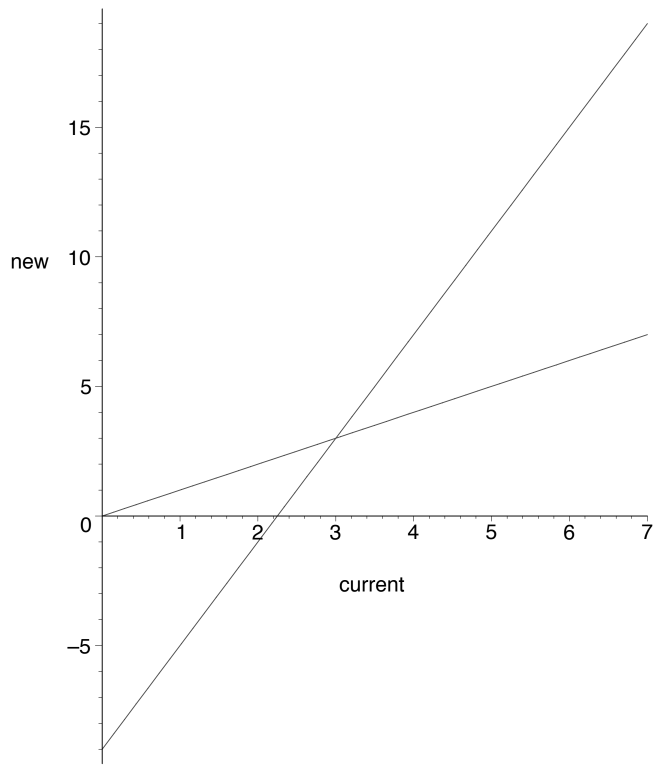


Figure 2: Graph of $a_{n+1} = 4a_n - 9$

Example 5. Discuss the long-term behavior of the sequence defined by $a_{n+1} = a_n^2$ with the initial condition $a_1 = \frac{1}{2}$.

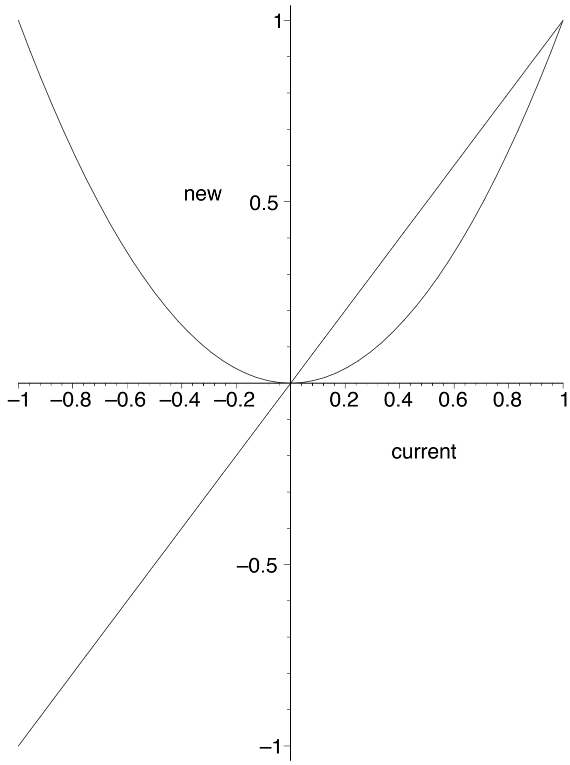


Figure 3: Graph of $a_{n+1} = a_n^2$

PROBLEM SET 2

1. Discuss the long-term behavior of the sequence defined by $a_{n+1} = 4a_n - 9$ with the initial condition $a_1 = 3$. Do the same for $a_1 = 0$.

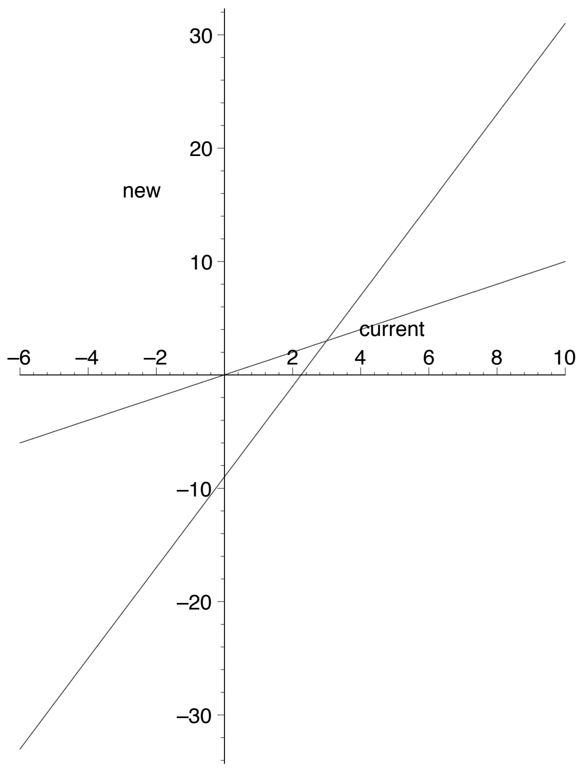


Figure 4: Graph of a_{n+1}

2. Discuss the long-term behavior of the sequence defined by $a_{n+1} = a_n$ for any initial condition a_1.

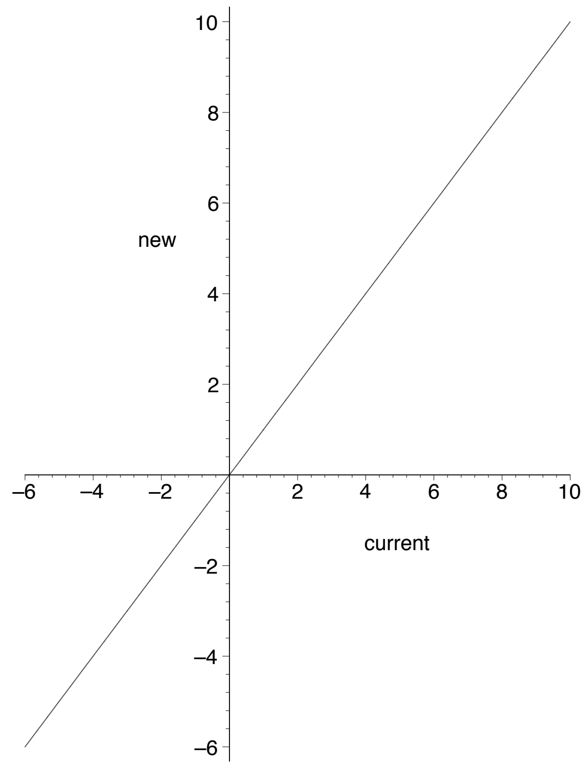


Figure 5: Graph of a_{n+1}

3. Discuss the long-term behavior of the sequence defined by $a_{n+1} = a_n^2$ with the initial condition $a_1 = \frac{11}{10}$; Do the same for $a_1 = 1$, $a_1 = \frac{9}{10}$, $a_1 = -\frac{1}{2}$, and $a_1 = -3$.

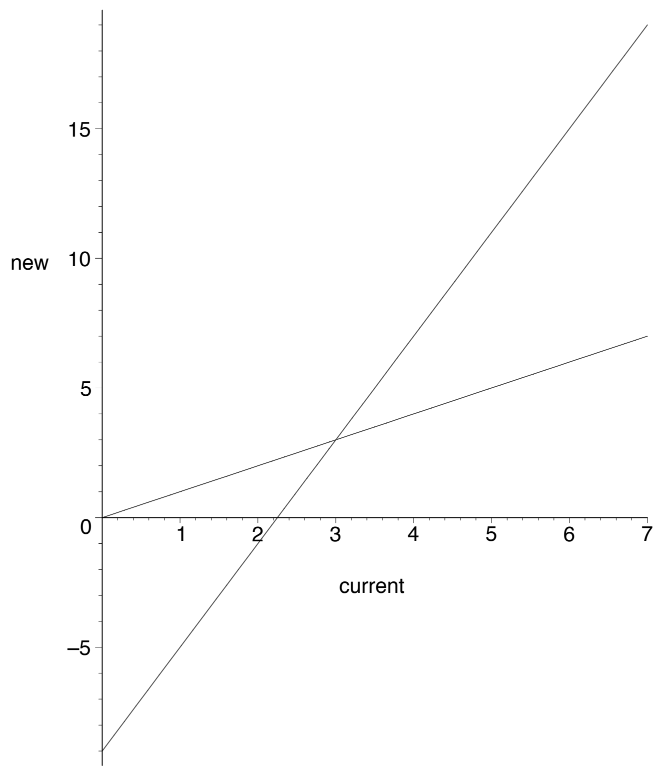


Figure 6: Graph of a_{n+1}

4. Exploratory Exercise. Discuss the long-term behavior of sequences defined by $a_{n+1} = a_n^3$ with the initial condition a_1. The goal here is to choose some initial conditions and see what happens. Try to expose all types of long-term behavior this recrusion relation exhibits (depending on initial conditions of course).

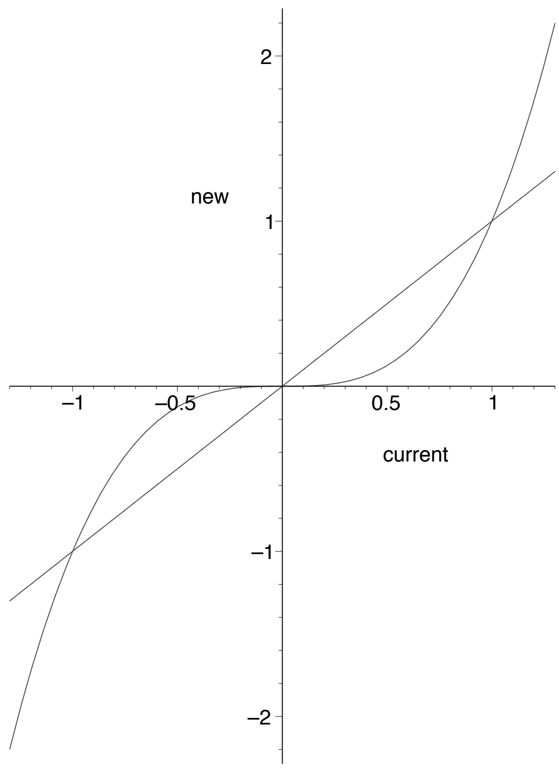


Figure 7: Graph of a_{n+1}

5. Exploratory Exercise. Do the same thing for $a_{n+1} = a_n e^{1-\frac{a_n}{10}}$ with the initial condition a_1.

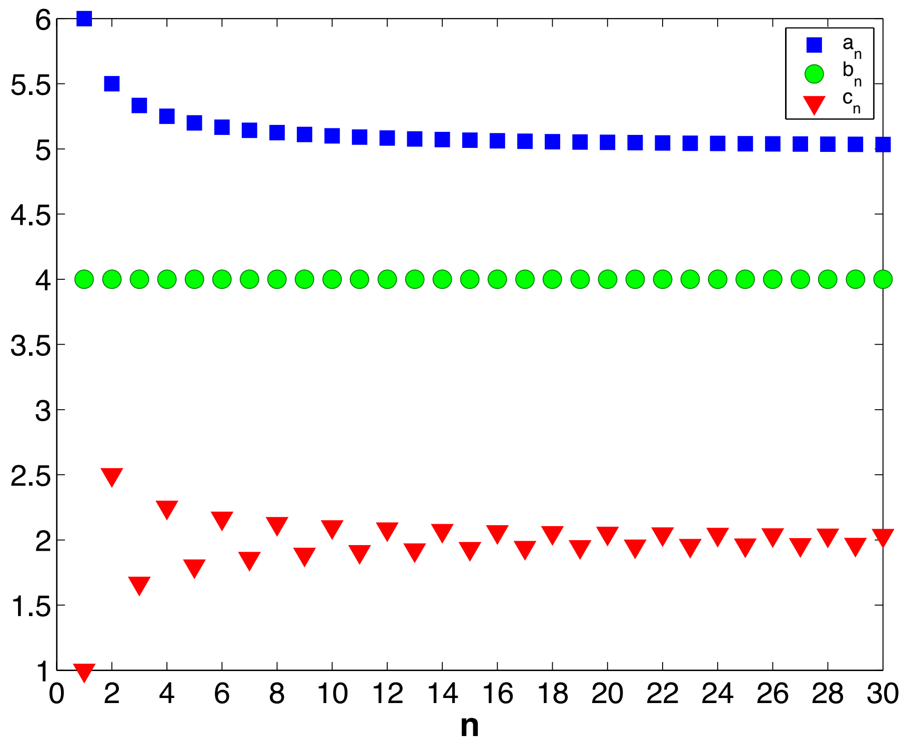


Figure 8: Graph of $a_{n+1} = 4a_n - 9$

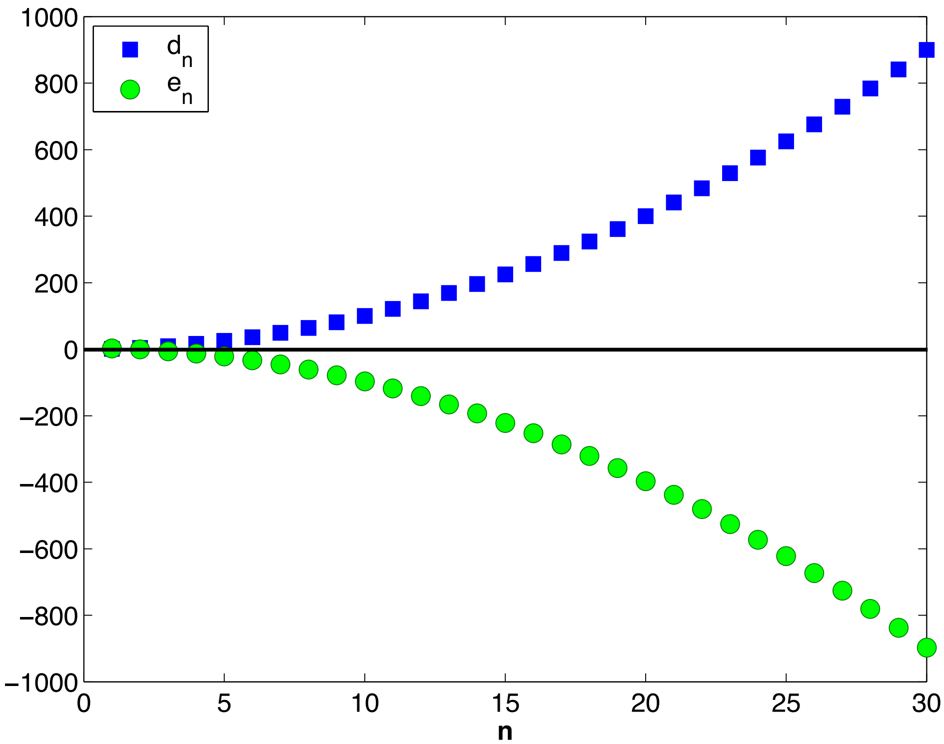


Figure 9: Graph of $a_{n+1} = 4a_n - 9$

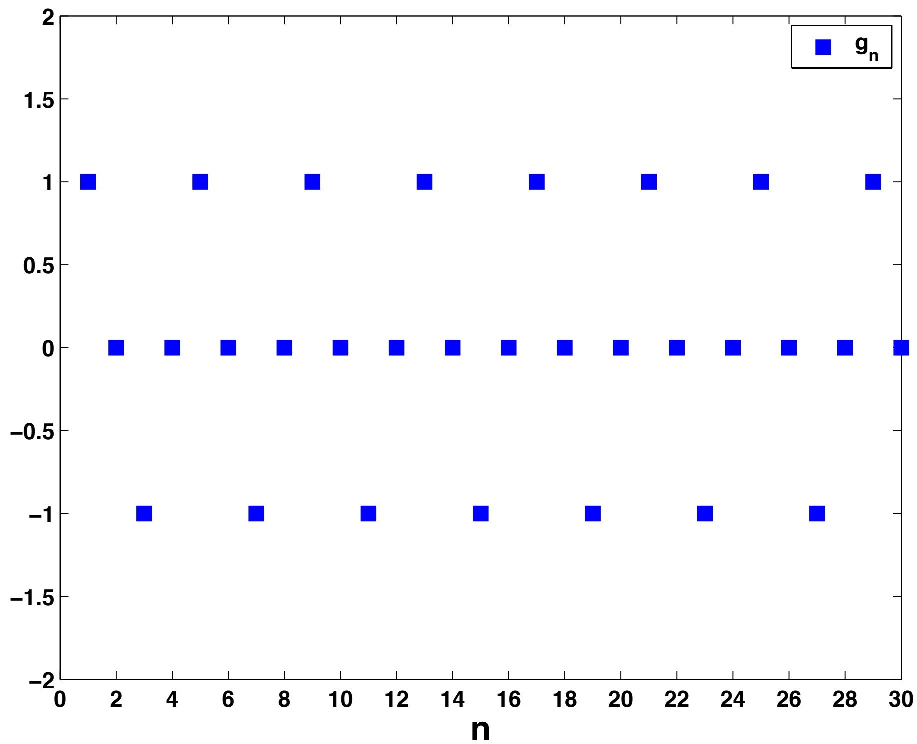


Figure 10: Graph of $a_{n+1} = 4a_n - 9$

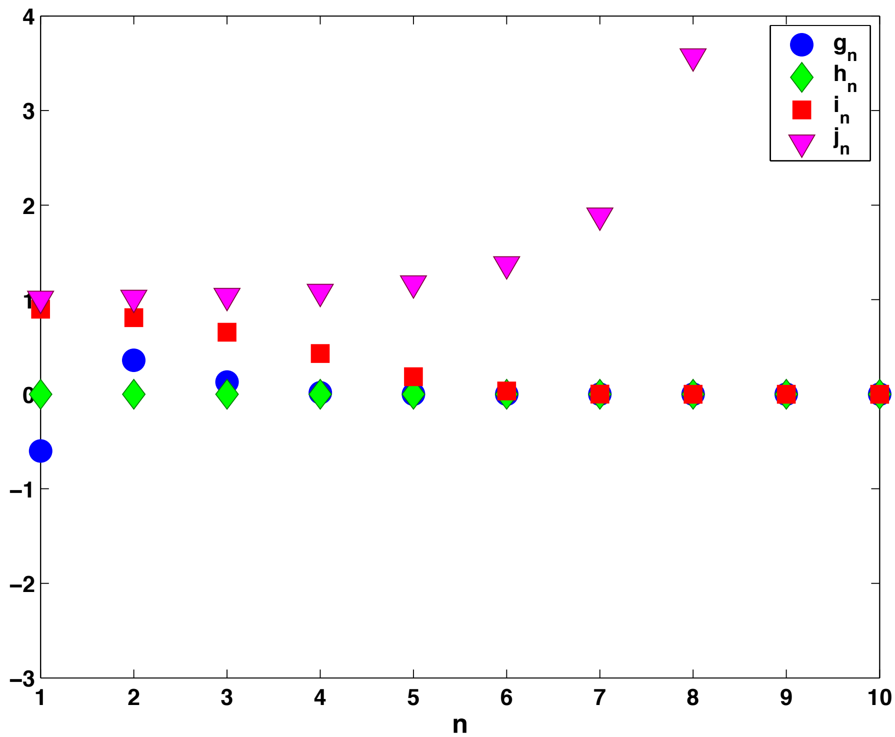


Figure 11: Graph of $a_{n+1} = 4a_n - 9$