

# Plato and Euler

## Platonic Solids

Geometric figures in the plane, composed of straight lines, are called polygons. Common and familiar examples include triangles, squares, rectangles, trapezoids, quadrilaterals, pentagons, hexagons, octagons, etc. There is a 3-dimensional analog of polygons. A closed three-dimensional figure whose boundary is composed of planar sections is called a **polyhedron** (plural **polyhedra**). Note that each of these planar sections (called a **face**), is itself a polygon.

Common and familiar examples of polyhedra include cubes, boxes, pyramids, prisms, etc. In the plane, when a polygon is such that all of its sides are of equal length, and all of its angles are of equal measure, it is said to be **regular**. Likewise, a polyhedron is **regular** if all of its faces are congruent regular polygons, and the angles between adjacent faces are all equal measure.

Here are a couple of examples:

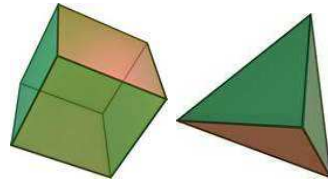


Figure 1: Cube, and Tetrahedron

Regular Polyhedra are often called Platonic Solids because Plato referred to them in his dialog, *Timaeus*.

*Question 1.* How many different Platonic Solids are there? Can we prove it?

## Counting Faces, Edges, and Vertices

We now know there are exactly five Platonic Solids. Here they are:

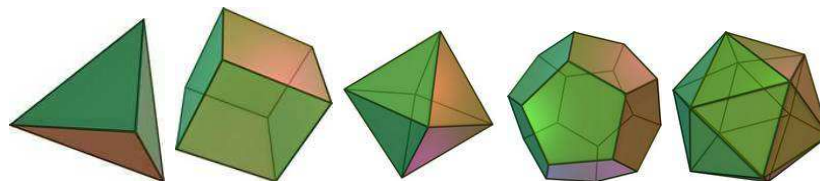


Figure 2: Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron

In proving that there are exactly five Platonic Solids, we counted the number of faces around a vertex. Let's make a table enumerating how many faces, edges, and vertices are in each of the five Platonic Solids.

Platonic Solid	Faces	Edges	Vertices
Tetrahedron	4		
Cube	6		
Octahedron	8		
Dodecahedron	12		
Icosahedron	20		

Table 1: Platonic Solids: number of faces, edges, and vertices.

*Question 2.* Fill in the rest of the table. We don't have these objects in front of us, but you can try to visualize them. For the dodecahedron and the icosahedron, you can ask yourself questions like "how many edges are in each face?" to help fill in the blanks.

*Question 3.* What patterns do you notice with the table above?

## Duality

In constructing the table for these Platonic Solids, we saw that the number of edges in the Cube and the Octahedron are the same. We also saw that the number of faces in the Cube is exactly the number of vertices in the Octahedron, and vice versa. We saw the exact same relationship between the Dodecahedron and the Icosahedron. Why?

*Question 4.* Draw a cube on your scratch paper. Now put a small filled square in the center of each of its faces. Connect the squares with lines if the two corresponding faces touch each other along an edge. What shape did you draw?

*Question 5.* Repeat the previous question, but start with an Octahedron. What do you get?

## More Duality

The Cube and the Octahedron are related in a special way, which we saw in the previous two questions. This relationship is called **geometric dual** (or sometimes just **dual**). In other words, we would say that the Cube is the dual of the Octahedron. You can probably guess by looking at the table we created that the Dodecahedron and the Icosahedron are duals to one another.

*Question 6.* If the object  $B$  is the dual of object  $A$ , is it necessarily true that object  $A$  should be the dual of object  $B$ ?

*Question 7.* What is the dual of the Tetrahedron?

## Euler Characteristic

We can also see something special in the table if we look along any row.

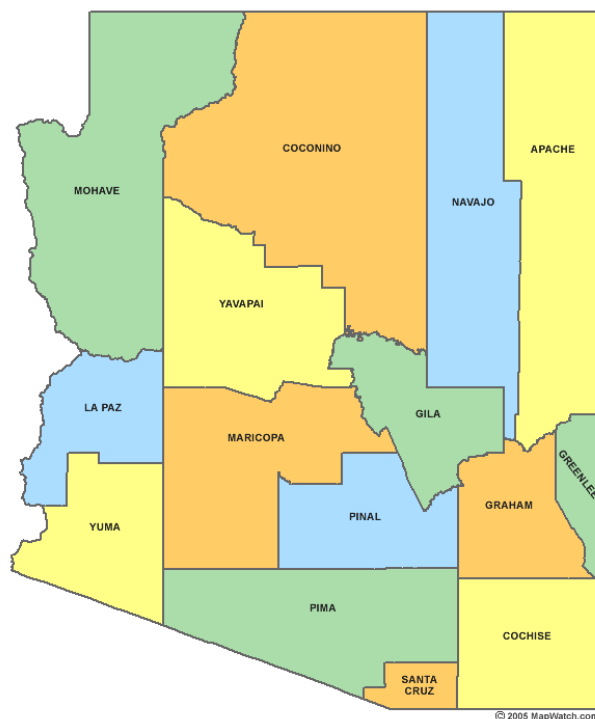
*Question 8.* Add the number of Faces to the number of Vertices, and Subtract the number of Edges. Do this for each row. What do you notice?

This number is called the Euler characteristic, and we will denote it  $\chi$ . Is it always the same?

*Question 9.* Try finding the Euler characteristic to other 3-D shapes. Consider familiar shapes like pyramids and prisms. What do you notice?

Is there anything in the definition of Euler characteristic that requires the edges or faces to be straight or flat?

*Question 10.* Below is a map of Arizona and its counties. Treat each county as a face; treat their boundaries as edges; and where two or more edges come together, treat that point as a vertex; also treat everything in the plane outside of Arizona as a face as well. What is the Euler characteristic of this map? Why is this the case?



So far, everything we have looked at really is being done on a sphere (Why?).

*Question 11.* What is the Euler characteristic of a sphere?

I claim that Euler characteristic is an invariant that depends only on the topology of the object itself. Therefore, it doesn't change if we bend, stretch, or compress the object. Moreover, if we added vertices, edges, or faces to the object, without changing its topology, then the Euler characteristic will remain the same. To see this, let's first look at the formula for Euler characteristic again:

$$\chi = F - E + V.$$

*Question 12.* What happens if you add a vertex to an object? Does the number of edges or faces change?

*Question 13.* In what ways can we add an edge to an object? What do these do to the Euler characteristic?

*Question 14.* In what ways can we add a face to an object? What do these do to the Euler characteristic?

So the Euler characteristic is a number intrinsic to the underlying topology of an object, not its specific geometry. Now let's see if the Euler characteristic can ever be a "non-two" number.

*Question 15.* What is the Euler characteristic of the Torus?

*Question 16.* What is the Euler characteristic of the surface with 2 holes?

*Question 17.* Can you come up with a conjecture for the Euler characteristic for orientable surfaces with  $g$  holes?<sup>1</sup>

Non-orientable surfaces have computable Euler characteristics as well.

*Question 18.* What is the Euler characteristic of the Klein bottle?

*Question 19.* What is the Euler characteristic of the Projective Plane?

Also, we can talk about the Euler characteristic of surfaces-with-boundary. We just need to be careful and include all of the vertices and edges in the boundaries of the surface.

*Question 20.* What is the Euler characteristic of the Cylinder?

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<sup>1</sup>The number of holes in a surface is called its genus; that is the reason for denoting it  $g$ .

*Question 21.* What is the Euler characteristic of the Möbius Strip?

*Question 22.* Recall that the Klein bottle is the connected sum of two Möbius Strips. Do your Euler characteristic calculations for these objects make sense in light of this fact? Why or why not?

*Question 23.* Similarly, the genus-2 surface is the connected sum of two genus-1 surfaces. Does our conjecture above make sense in light of this fact? If we have two surfaces,  $S$  and  $T$ , and we know what their Euler characteristics are, say  $\chi(S)$  and  $\chi(T)$ , what is the Euler characteristic of the connected sum  $S\#T$ ?

*Question 24.* There is another way to join two surfaces,  $S$  and  $T$ , namely to glue them together at one point<sup>2</sup>; the fancy word for it is the **wedge product**, and is denoted  $S \wedge T$ . What is the Euler characteristic of  $S \wedge T$ , given the Euler characteristics of  $S$  and  $T$ ?

*Question 25.* If two objects have the same Euler characteristic, must they be topologically equivalent? Can we come up with a bunch of different surfaces with the same Euler characteristic?

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<sup>2</sup>The resulting object is not a surface, but it is still an interesting object worthy of discussion