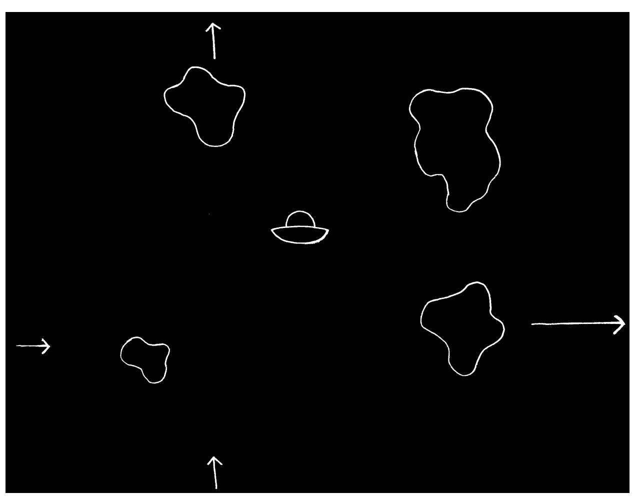
Asteroids: What is the shape of your Universe?

Introduction

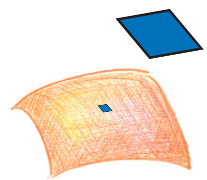
In the computer game *Asteroids*, UFO spaceships, and asteroids are flying across the screen. Wherever an object hits the boundary of the screen, it comes out from the opposite side, as shown by the arrows in the figure below. The question we want to answer is: "What is the shape of the Universe in this game?" From there, we will procede to more bizarre Asteroid games and investigate the geometry of many possible universes. First some geometric preliminaries:



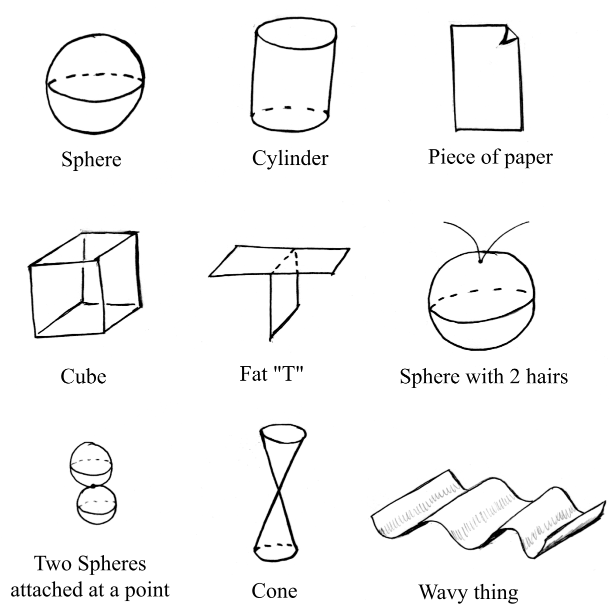
Surfaces

We all have an intuitive concept of a “surface” as of something 2-dimensional, some sort of object we can obtain by deforming a piece of paper or a rubber sheet. For our purposes, we won’t need much more sophistication than that.

Definition. A ***surface*** is a geometrical object such than at every point P, provided that we “zoom in” enough, it looks like a little piece of a plane.

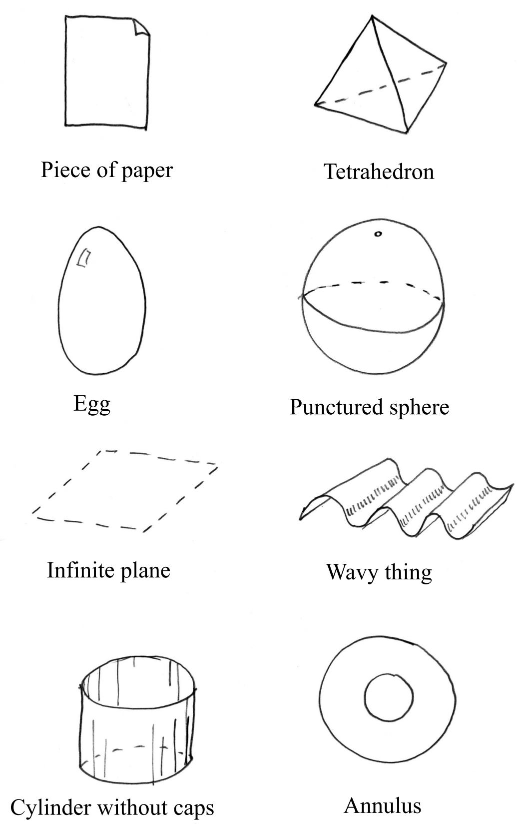


Question. Which of the following objects are surfaces? Which are not? Why not?



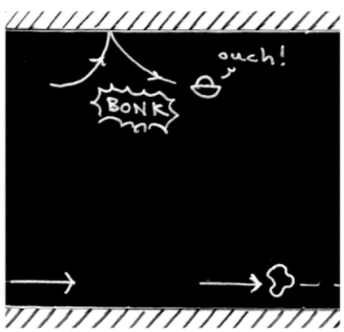
All of our surfaces will be made of some elastic, stretchable and bendable substance. Therefore, we will consider two surfaces to be the same if they can be stretched, pushed, and pulled one into the other. We'll find no difference between a perfect sphere, an egg, or even a cube! But beware: no pinching or squishing is allowed.

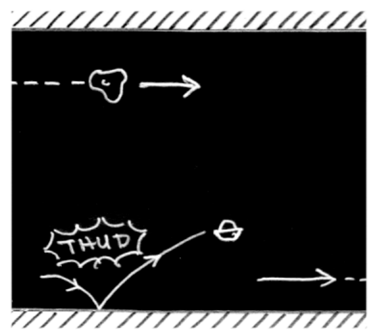
Question. Which of the following surfaces are equivalent?



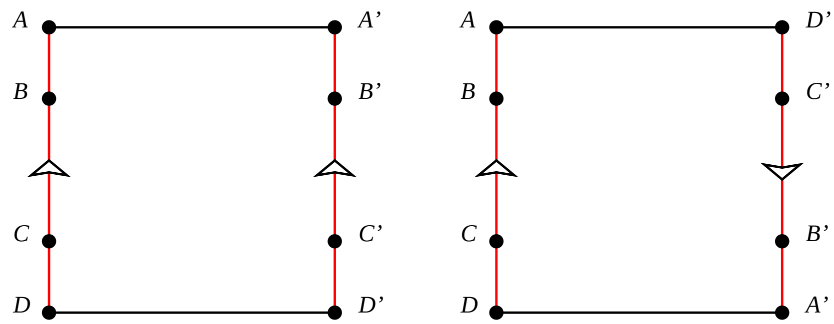
Asteroids

Now let's return to our *Asteroids* Universe and look at two easier cases. In each, the top and bottom boundaries are solid; *i.e.*, anything hitting them will simply bounce back. The vertical sides will behave as indicated in the pictures.



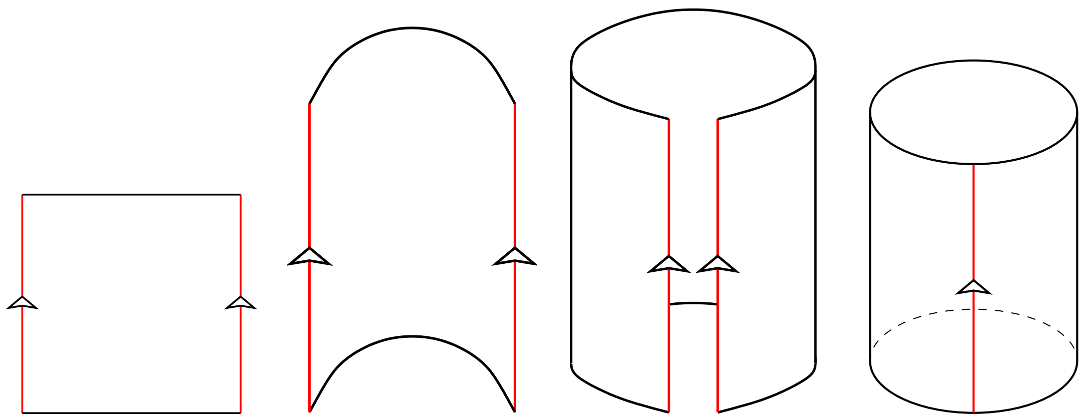


In case (a), the asteroid disappearing on the right at a certain height will reappear on the left at the same height. In case (b), the objects going out on one side at a certain distance x from the bottom will reappear at the same distance from the top! So in the pictures, the point A and the point A' are actually the same point. As are the pairs B, B' and C, C' and so on. Furthermore, the righthand and lefthand sides of the screen are actually the same line. Think of the sides being "glued" together. We will denote this identification with arrows, and we will glue the edges together so that the arrowheads point in the same direction after the glueing.

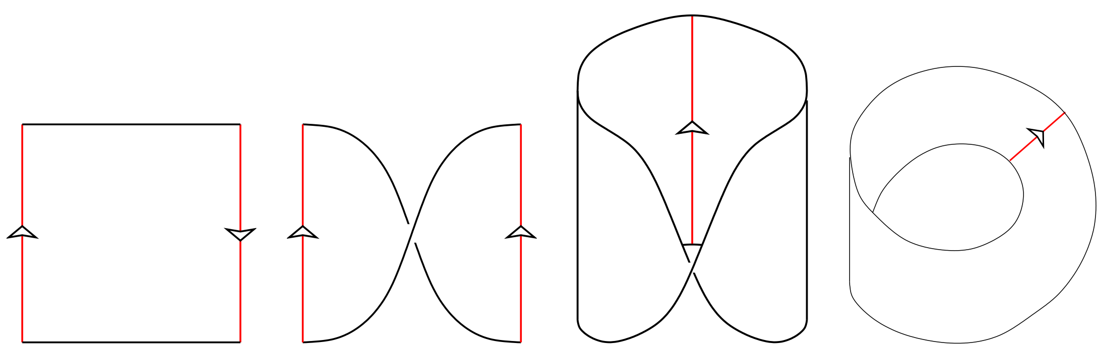


Question. So what are the shapes of these universes?

The answer is obtained by physically stretching and bending our universes until we can really glue the sides together that we want to identify:



This is the familiar cylinder.

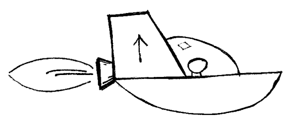


This is a more bizarre object which we will call the *Möbius Strip*. Make one out of paper and scotch tape!

Silly Question. How many sides do our surfaces have? Duh! Two, of course! A front side and a back side, right? Are you sure? I agree that the familiar cylinder has two sides, but have you tried taking a stroll along the Möbius strip? (Do so!)

Surprise! We have in our hands an example of a one-sided surface. Being able to discern the one- or two-sidedness of our Universe seems heavily dependent upon the physical realization of our surface. Is there any way to tell how many sides our Universe will have just by looking at the square with the sides identified?

The answer is yes! And we'll have Spaceman Spiff in his spaceship to help us figure out how! Let the fearless Spiff wander around freely in our universes and then come back to



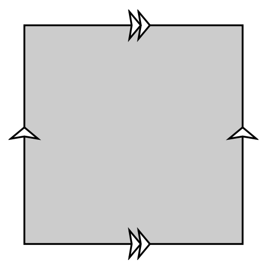
where he started.

Question. What happens in one case that can't happen in the other?

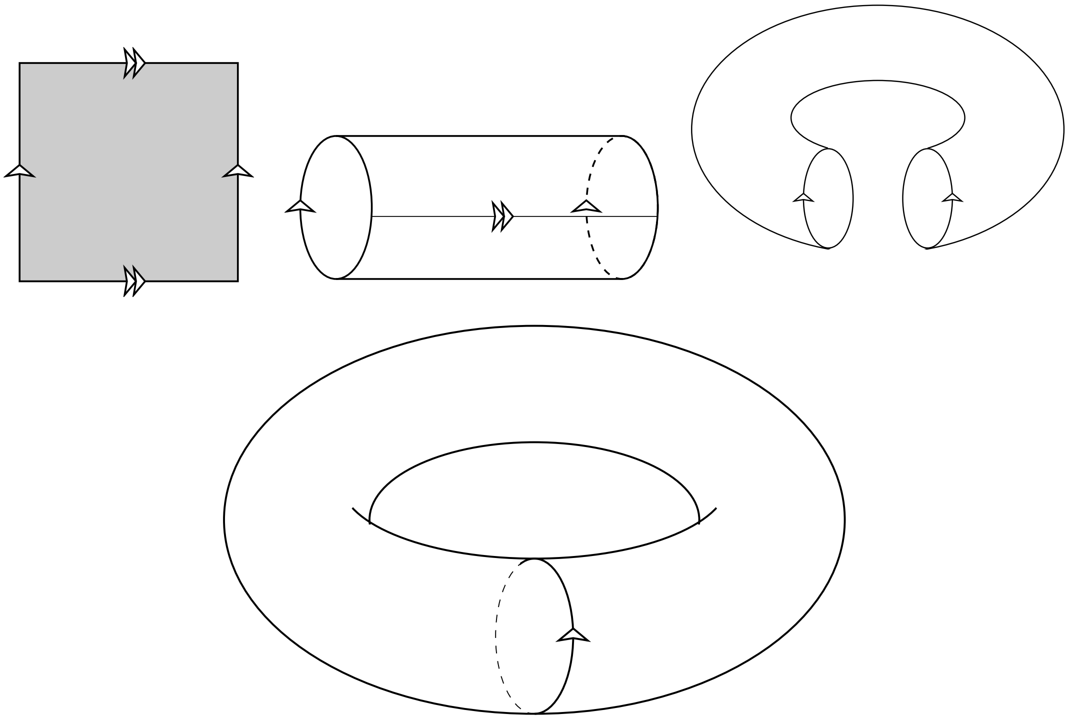
In the cylinder, every time Spiff returns home, if his spaceship is facing right, he will be heads up. Whereas, in the Möbius strip, it may very well happen that pointing the spaceship to the right, he will have ended up heads down! We will call this phenomenon SSRP (Spaceman Spiff Reversing Property), and we will say that a surface is ***non-orientable*** if it has the SSRP, and ***orientable*** if it does not.

Now on to more fun. We are now ready to explore new and more interesting universes. Let's go back to our very first universe,that we will represent with our new notation:

Question. What is the shape of this universe?

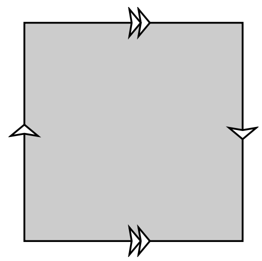


Indeed this represents a "donut"-shaped universe. Mathematicians call this shape a ***torus***.



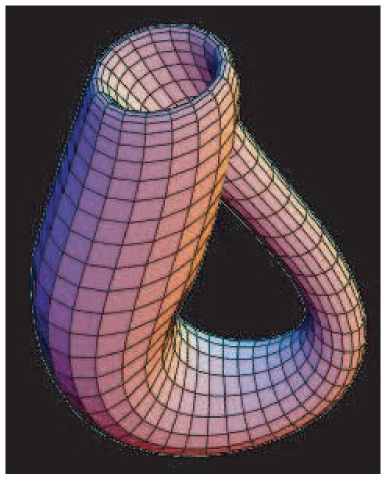
Question. Is the torus orientable or not?

Now let's complicate things a bit. Consider this universe:



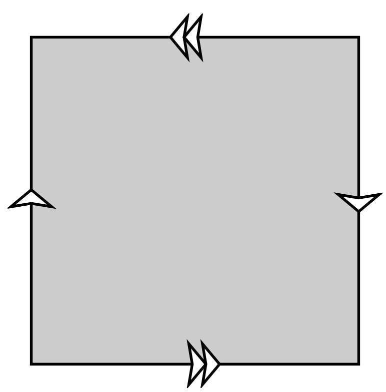
Question. What do you think this strange and alien universe looks like?

This bizarre thing is called a ***Klein bottle***, and there's not quite enough "room" in our 3-dimensional space for it to exist properly. The best we can do is to let it go through itself, to look something like this:



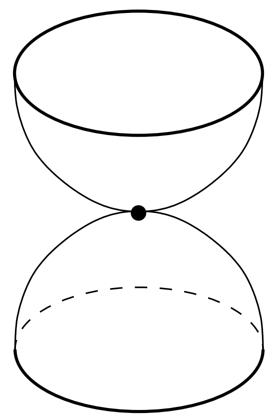
If we had four dimensions to work with, we could draw this without it passing through itself. You shouldn't think of the Klein bottle as actually intersecting itself, just that it is the best we can do when drawing it 3-dimensionally.

The next logical question would be to investigate this:



Question. Why not? What is wrong with this object?

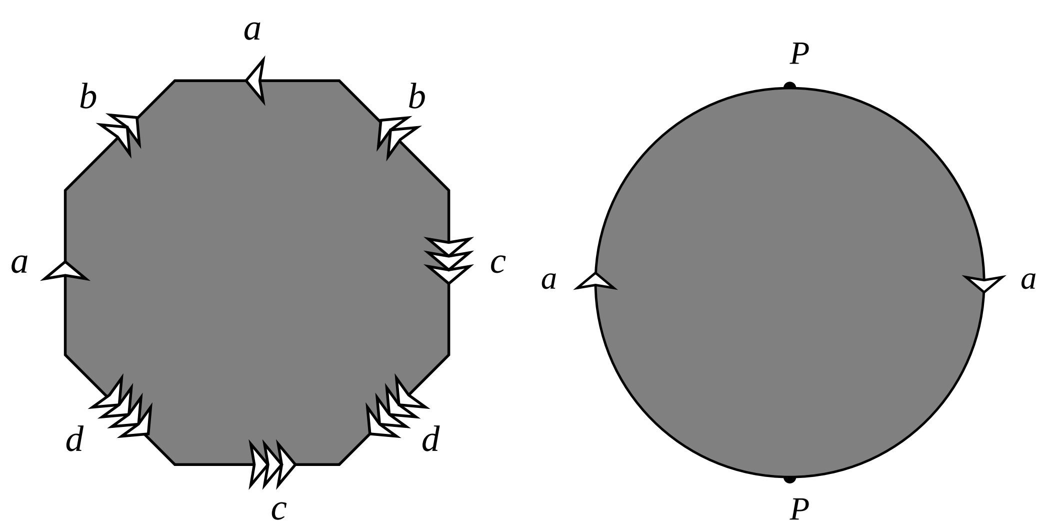
The above object is not a surface! It contains a point, which nearby the object looks like this:



Now it seems that the most familiar surface of all, the sphere, is not appearing at all! Let's then reverse the question:

Question. What sort of *Asteroids* game should I play so that the universe is spherical?

Now we can move on to yet more complicated (and fun!) *Asteroid* games. How about these:



Question. What do these surfaces look like?

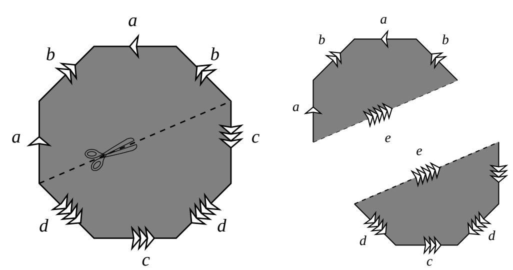
Now the first one is a “donut” with 2 holes, which we will call a ***genus-2 surface***, and denote it T_2. It is possible , but complicated, to see this just with the process of stretching and bending, and glueing.

Question. Try to do so!

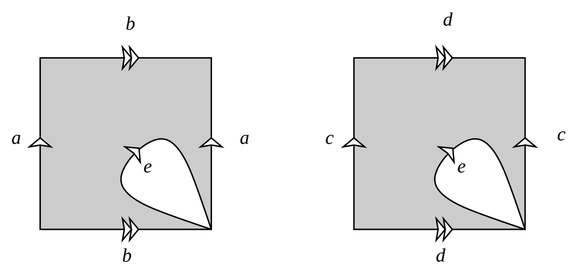
The second surface is called a ***projective plane*** and is denoted by $\mathbb{P}^2$. It’s not really easy to visualize, even thought it’s an extremely helpful and familiar friend to many mathematicians. There are many ways to develop a good insight and intuition on its nature. Let’s put it on the back burner for now, and we can come back to it.

Polygons

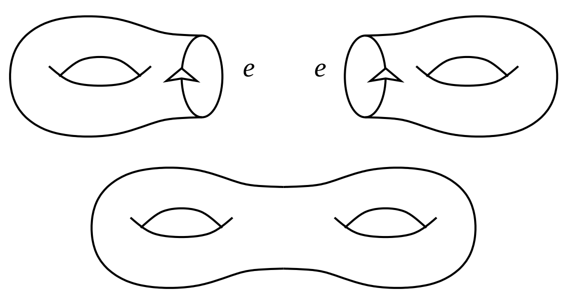
Let's look at the two-holed donut, T_2, once again. There is a trick that allows us to avoid going through all the trouble we went through with the previous exercise. Namely, we can cut it in half, remembering where we cut it,



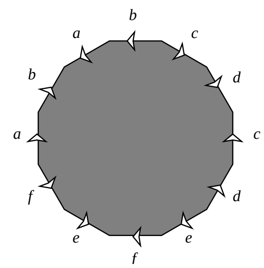
move it around, and look at the two pieces:



Notice that each of these is a torus with a disk missing. Therefore, they can be glued together along those circles (which are exactly the cut that we made in the first place):



Question. What is this object? Can you prove it?



Question. What does the next object in the sequence look like, as a polygon?

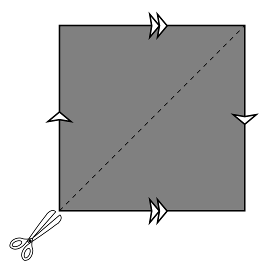
Classification of 2D Universes

We have now unknowingly introduced a very useful and interesting operation that allows us to relate and create new surfaces. Given two surfaces, it's always possible to remove a little disk from each of them and then glue them along the boundary of the "hole" to get a new surface.

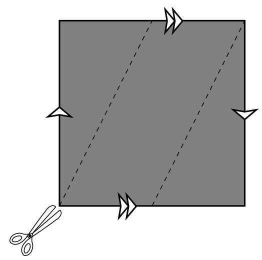
This operation is called ***connected sum***, and is denoted by the symbol #. So we already have seen that $T_2 = T_1 \# T_1$, and $T_3 = T_2 \# T_1 = T_1 \# T_1 \# T_1$. Let's see if we can find some others.

Let's try cutting the Klein bottle like we did for T_2.

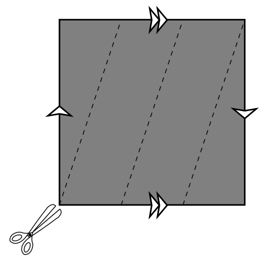
Question. What do we get if we cut the Klein bottle like this?



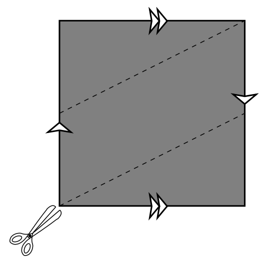
Question. What if we instead cut the Klein bottle like this?



Question. Or like this?



Question. Or this?



So we see that the Klein bottle can be thought of as the connected sum of two Möbius strips. Cool. What about $\mathbb{P}^2$?

Question. Can you figure out what two surfaces we have seen make up $\mathbb{P}^2$? What happens if you cut a disk out of it?

Can we classify all surfaces? We sure can.

Question. All 2D Universes can be classified by their type. Can you figure out what these types are? Is that all of them?