The fundamental group

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1 Why you should never hang a picture with a topologist

There are two nails in the wall. You have a picture with a long cord to hang it by. How long? As long as you want it to be! You want to hang the picture so that no matter which of the nails you pull out, the picture will fall. Why would you want to do that? Well, you're a topologist and you like symmetry.

Notice that hanging it in the standard fashion on both nails as described on the left below, simply won't do because if we pull out any one nail, the picture will continue dangling on the remaining one. The sketch on the right is not satisfactory either, since pulling out the left nail will not have any effect on the picture. So what do we do? Wait patiently, we'll find a solution (actually, many solutions) in section 6. But, maybe you'd like to try your hand at it right now?

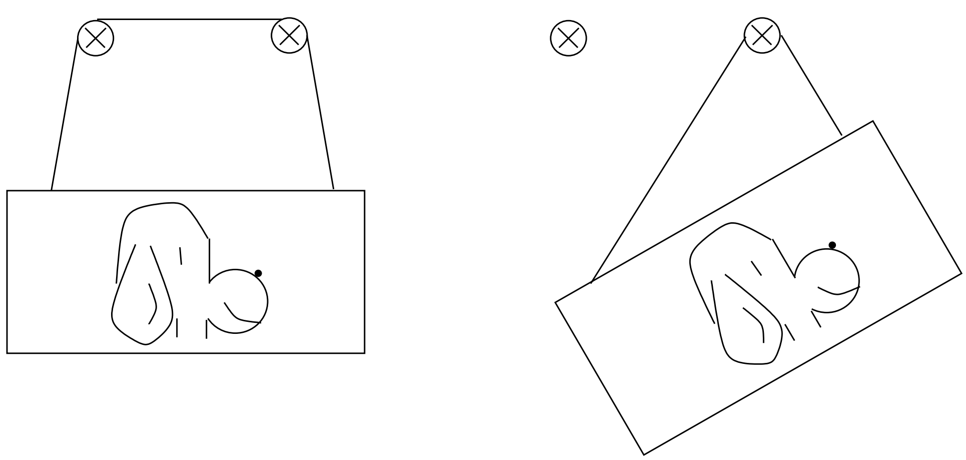

Figure 1: *Hang the picture so it will fall while pulling out any nail.*

2 Recap: Klein's Group and some notation

Recall that a group consists of reversible actions that we can carry out one after the other. Sometimes we call that composition: to compose action b with action a means first do a, then do b and is denoted: $b \circ a$. Yes it's confusing, you compose the second action with the first! In order not to get confused, we will refer to this sequence of actions as as multiplying a with b, and denote it $a \cdot b$ which is read left to right: first a then b.
Last time we used Klien's group to elegantly show that the octagon peg game was insolvable. Recall that Klein's group contained 4 elements: e - don't do anything, a - press button A which turns the left light bulb on/off, b - press button B which turns the right light bulb on/off, and c - press both A and B.
a is a reversible action, this means that we can do a and then its *inverse* and it will be the same as doing nothing at all. In K, a's inverse is a itself since pressing A twice is like doing nothing. We will denote this by $a^{-1} = a$. Similarly, $b^{-1} = b$, $c^{-1} = c$ and $e^{-1} = e$. For any action t we denote its inverse by t^{-1}. t^{-1} is the action such that $t \cdot t^{-1} = e$.

3 Boot camp and homotopy

Suppose we are all recruited to the army. Our camp is surrounded by a fence, there's a spot marked P, and a tree. The group we are about to describe will contain information of some geometric properties of our camp.

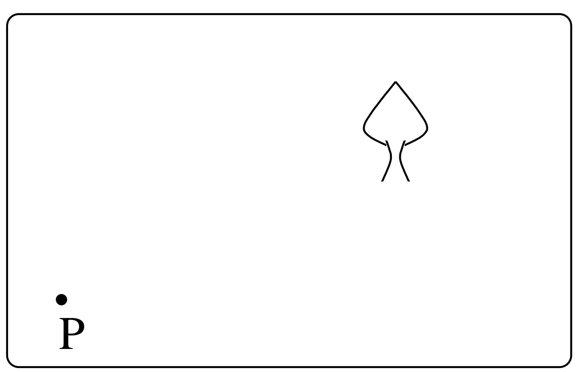

Figure 2: *A sketch of our camp*

An action t in our group G will be: walk along path T. Our paths will always start and end at the same point, P. $t \cdot s$ means walk along T and then walk along S, or equivalently, walk along the path created by joining T with S.

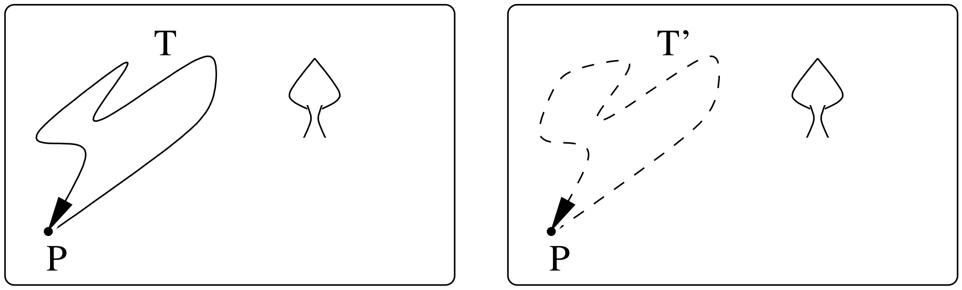

Figure 3: *Paths T and T' are very close so will will not distinguish between them.*

The path T is very close to path T', and we are not very picky, so we will consider walking along path T and walking along T' as the same action.
This seems reasonable, but there's a heavy price to pay for our slackness. Suppose the officer commands one of his soldiers to run up to the tree and back. The commander shouts out "z", which means, run along path Z.

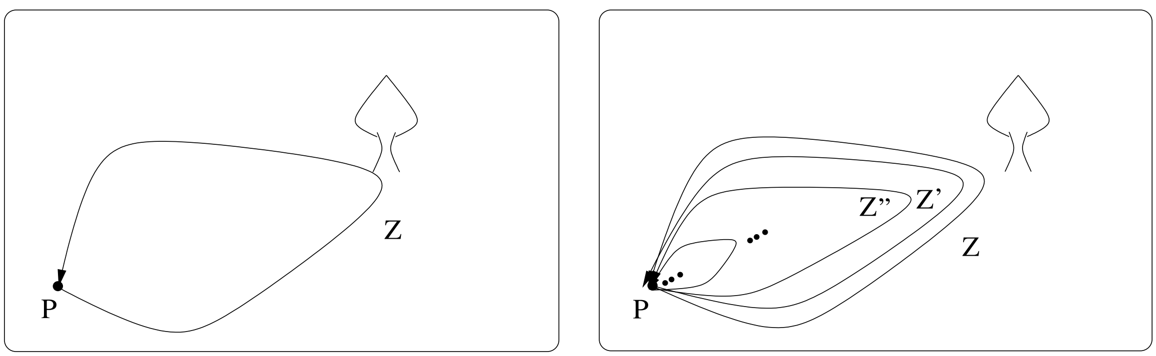

Figure 4: *Z goes up to the tree and back, but is actually the same as staying at point P.*

But the soldier never stirs from his place. The officer surprised at this blatant show of disobedience asks the soldier why is he not moving. The soldier then replies that path Z is very close to path Z' and path Z' is very close to Z'' and so on. "So", the soldier philosophizes, "running along path Z is the same as staying in place. Sir!"

It seems that our initial lack of distinction between T and T' has made many of our paths equivalent to a point. Traversing T, T' and Z are all equivalent to staying in place (at point P). Maybe all of our paths are equivalent to staying in place?

No! Circling the tree will not be equivalent to staying in place. The reason is that if we want to shrink path A in our sketch, as we shrunk path Z, we'd have to go through the tree which is not allowed. Let's think about this for a moment A is equivalent to other paths such as A' and A'' but not to path Z or to the point P. In mathematical terms, we'll say

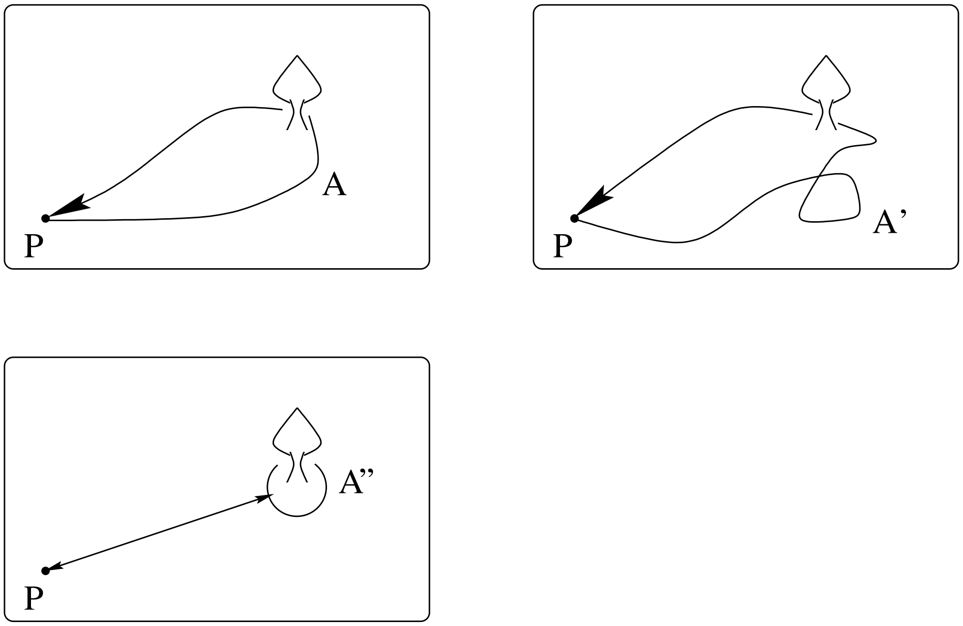

Figure 5: *Path A circles the tree.*

that A is homotopic to path A', since we can continuously change A to A' (i.e. change A to A' in very very small steps). Path Z is homotopic to point (or path) P, but path A is not homotopic to P.

Now the officer can safely order "a! on the double!" and the soldier can do nothing but comply.

4 The fundamental group

An action in the fundamental group G is travelling along a path in our camp where we don't distinguish homotopic paths. We have: e - stay in place (do nothing), and a - walk path A. What is $a \cdot a$? It means, walk A twice, or circle the tree twice, and we shall denote it a^2.

So G contains $e, a, a^2, a^3, \ldots$ where a^n means circle the tree n times. But what is the inverse of a? We need to find some path B so that walking along A and then B will be the same as staying in place. It seems impossible! But this is precisely where our indistinction of small changes comes into play! We would like that a^{-1} will be - walk A in the other direction. $a \cdot a^{-1}$ would then be traversing the path in the sketch. But we can homotope this path to a point (we are allowed to leave point P in the middle of the path as long as we start and end at P).

Now we can honestly write $a \cdot a^{-1} = e$. Similarly, for any path T if t is traversing T in

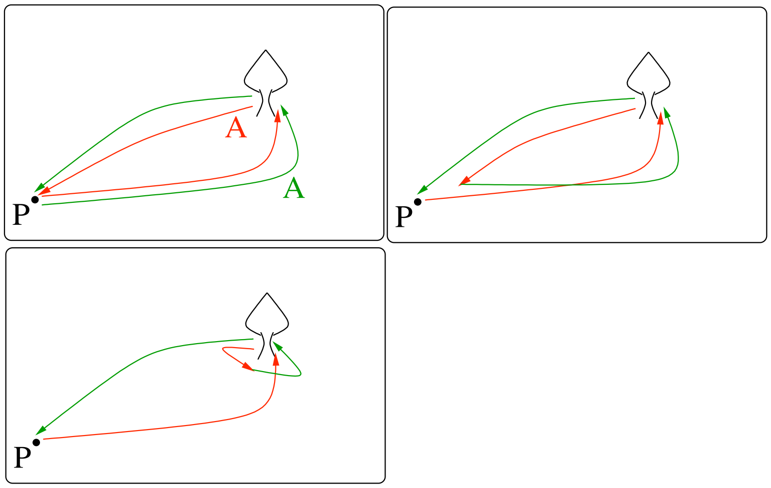

Figure 6: *traversing this path is* $a \cdot a = a^2$.

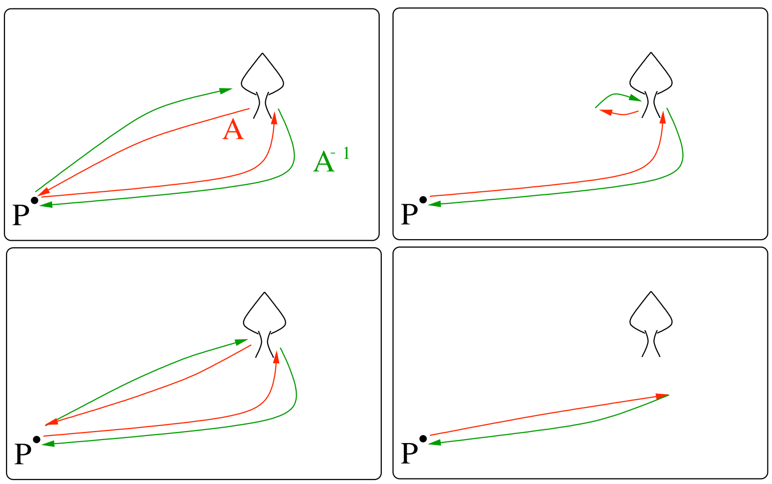

Figure 7: *This path is homotopic to a point.*

one direction, t^{-1} is traversing it in the other direction.

The fundamental group of our camp is $G = \{\ldots, a^{-2}, a^{-1}, e, a, a^2, \ldots\}$ with the multiplication rule: $a^n \cdot a^m = a^{n+m}$ for every pair of integers n and m.

But what properties does this group measure? Does it take into account the shape of the camp? What the starting and ending point was chosen to be? The tree in the camp? We'll be in a better position to answer these questions after considering a camp with two trees.

5 Two trees

Now suppose that our camp has two trees, as in the figure 8.

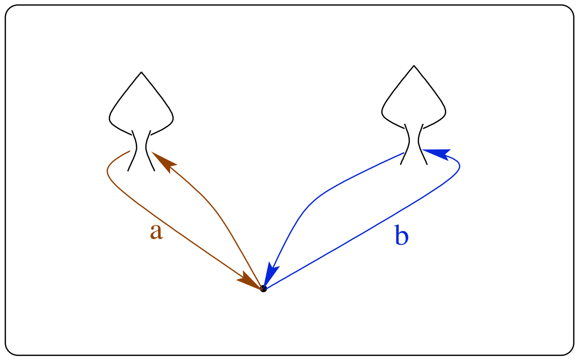

Figure 8: *A camp with two trees.*

Our fundamental group (traversing paths) will contain:

$$\ldots, a^{-1}, e, a, a^2, \ldots, b^{-2}, b^{-1}, b, b^2 \ldots$$

What other elements are there in the group? See figures 9, 11, 12

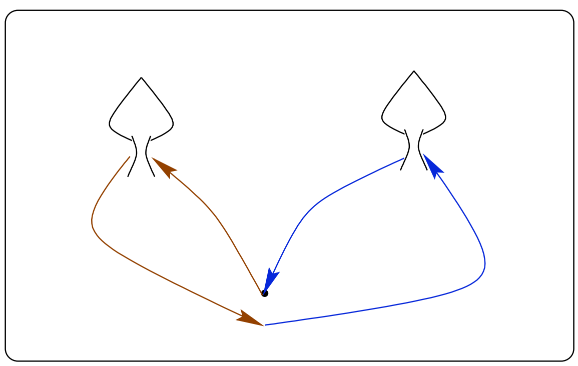

Figure 9: $a \cdot b$

There is a correspondence between paths in our camp and words in the letters: a, b, a^{-1} and b^{-1}. A word that doesn't contain $a \cdot a^{-1}$, $b \cdot b^{-1}$, $a^{-1} \cdot a$ or $b^{-1} \cdot b$ is called *reduced*. If a word w contains one of these then the corresponding path must contain backtracking. Omitting the backtracking in our path is tantamount to deleting the surplus $a \cdot a^{-1}, \ldots$ in our word.

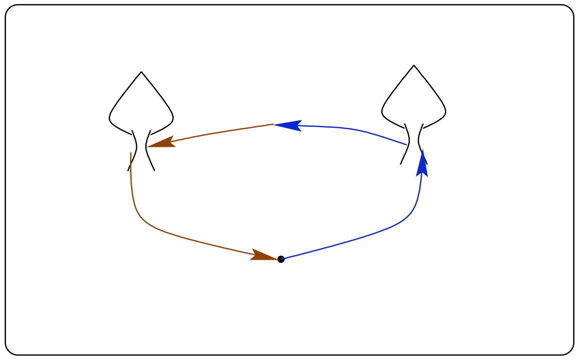

Figure 10: $b \cdot a$

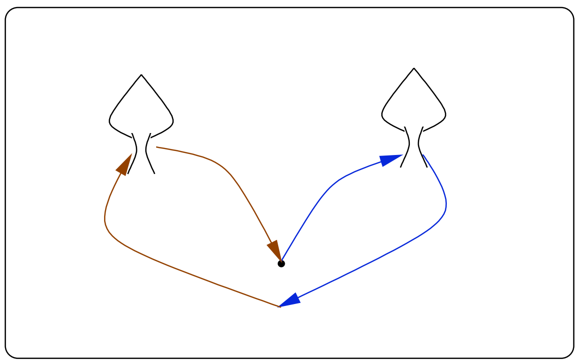

Figure 11: $b^{-1} \cdot a^{-1}$

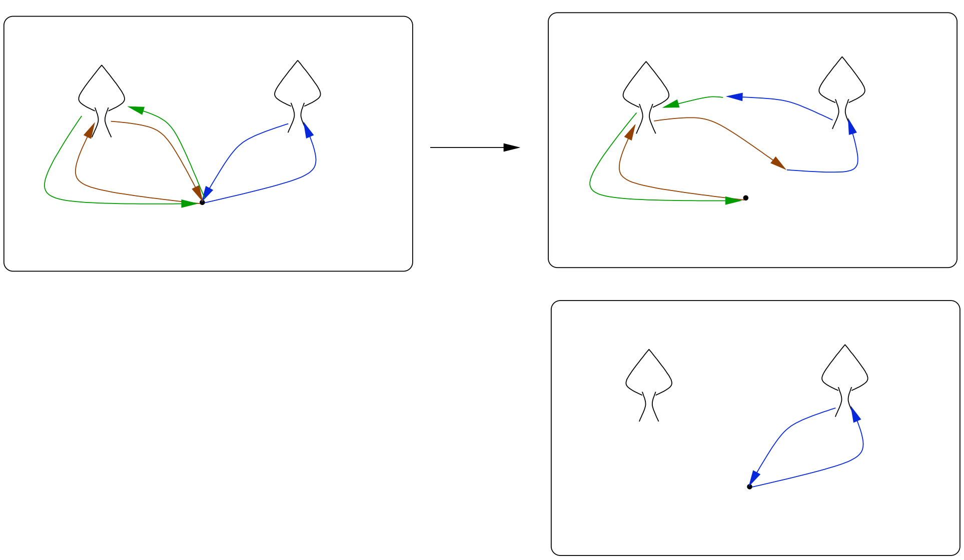

Figure 12: $a^{-1} \cdot b \cdot a$ *is not equivalent to* b

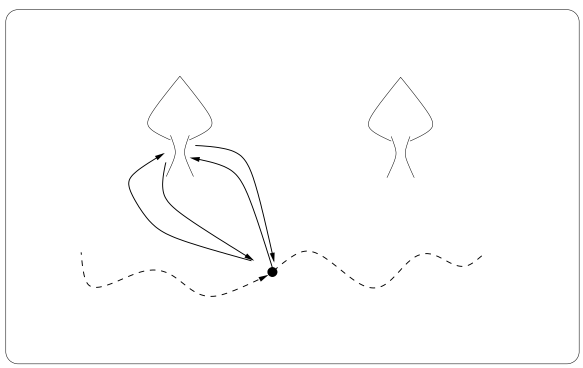

Figure 13: *We can homotope $a \cdot a^{-1}$ to a point.*

6 Hanging the picture

The bare wall with two nails is analogous to our camp with two trees. Winding the cord around the nails corresponds to a path in our camp which, in turn, coins a word in the letters: a, b, a^{-1}, and b^{-1}. The laws of nature will ensure that w will always be reduced (no $a \cdot a^{-1}$ will appear). Now lets fix a word, say $w = a^2 \cdot b^{-1} \cdot a^{-1} \cdot b^3$. What happens to this word when we pull out a nail, say the left one? Every time we do a — it will be equivalent to doing nothing — there is nothing to hold the string. So all appearances of a and a^{-1} in w will change to e and w will change into $e^2 \cdot b^{-1} \cdot e^{-1} \cdot b^3$ which equals to b^2. If we pluck out the right nail all the bs and b^{-1}s will convert to e and w will change to $a^2 \cdot e^{-1} \cdot a^{-1} \cdot e^3$ which equals a.

Therefore, a solution to our riddle corresponds to a reduced word $w \neq e$ which will turn to e when we pull out either nail. For example, $w = a \cdot b \cdot a^{-1} \cdot b^{-1}$ is is a solution, see figure 14 (notice what happens when we change every a to e or every b to e). If w is any solution to our riddle, how does the number of as compare to the number of a^{-1}s? How does the number of bs compare to the number of b^{-1}s?

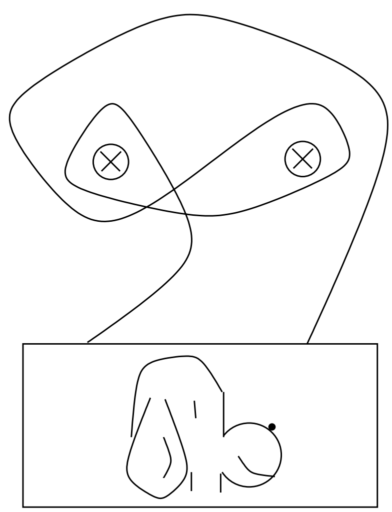

Figure 14: *A solution to the riddle*

The element $a \cdot b \cdot a^{-1} \cdot b^{-1}$ has a special form. It is called a commutator and denoted $[a, b]$. Any commutator will give a solution to our problem, for instance $[a \cdot b \cdot a, b^2] = a \cdot b \cdot a \cdot b^2 \cdot (a \cdot b \cdot a)^{-1} \cdot b^{-2} = abab^2a^{-1}b^{-1}a^{-1}b^{-2}$. The converse is not true. Is a product of two commutators a commutator? Is it a solution to our riddle?

7 More than two nails

Now that we found many solutions to our riddle, we can try to solve the same thing for 3 nails. As before, we'd like to wind the cord around the three nails so that whenever we take out a nail, the picture will fall.

Twisting the cord about the three nails will correspond to a word in the letters a, b, c, a^{-1}, b^{-1} and c^{-1}. Whenever we pull out a nail we convert all appearances of a, b or c to the trivial element e. We'd like to find a word w which becomes trivial when we pluck out a nail. Will the previous solution work for us now? No, since nothing will happen when we remove the nail marked c.

Not surprisingly, our solution will come in the form of commutators:

$$[[a,b],c] = (aba^{-1}b^{-1})c(aba^{-1}b^{-1})^{-1}c^{-1} = aba^{-1}b^{-1}cbab^{-1}a^{-1}c^{-1}$$

Try it! Can you solve for 4 or 5 nails?

8 What is the fundamental group measuring?

We've seen that the fundamental group for the camp with one tree is generated by a (this means that all elements are of the form a^n for some integer n). The fundamental group of a camp with two trees, is generated by a and b and is called the free group on two letters. In very coarse terms, the fundamental group measures the number of holes in our space. But it does so in a sophisticated way — using an algebraic animal — a group! A disc with two holes will NOT have the same fundamental group as a surface of a donut (torus) even though they both have two holes! However, the fundamental group will not be able to detect other geometric properties of the space. It will be the same for a circle and a square and a triangle. It will be the same for a sphere, a surface of a cube, and the surface of a pyramid.

The kind of coarse geometric properties that the fundamental group measures are called topological properties. The branch of math that connects topological concepts with algebraic ones is called "algebraic topology". These concepts had naturally arisen in the study of complex functions, but have since taken a life of their own and have proven to be very fruitful.

9 Exercises:

1. Which of the paths described in figure 15 is homotopic to a point? Sketch the shrinking process (homotopy).

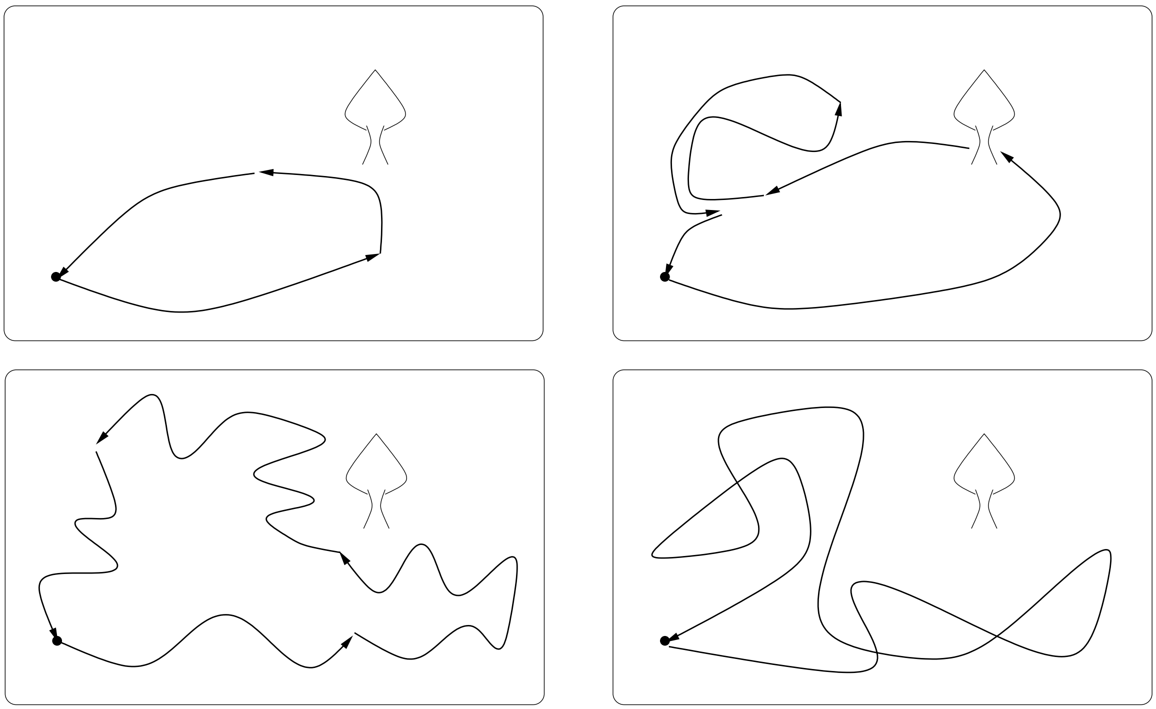

Figure 15: *Which of these is homotopic to a point?*

2. Which of the paths described in figure 16 are homotopic to each other?

3. Traversing each path in the previous question is the same as a^n for some integer n, where a is the circling the tree once counter-clockwise. Find n in each case.

4. For each word a, a^{-2}, a^5, a^{-3} sketch the corresponding path.

5. Sketch the homotopy that proves that $a^2 \cdot a^{-2} = e$. In other words, show that the path that circles the tree twice counter-clockwise and then circles the tree twice clockwise is homotopic to the point P.

6. For each path in figure 17 match the corresponding word.

7. Draw the corresponding paths to the words: $a, b^{-1}, e, a \cdot b^{-1}, a \cdot b \cdot a^{-1}$.

8. Write explicitely what $[a \cdot b^2, e]$ is. Do the same for $[e, a^3 \cdot b^2]$. Can you generalize? For any word w in a, b, what is $[w, e]$? what is $[e, w]$?

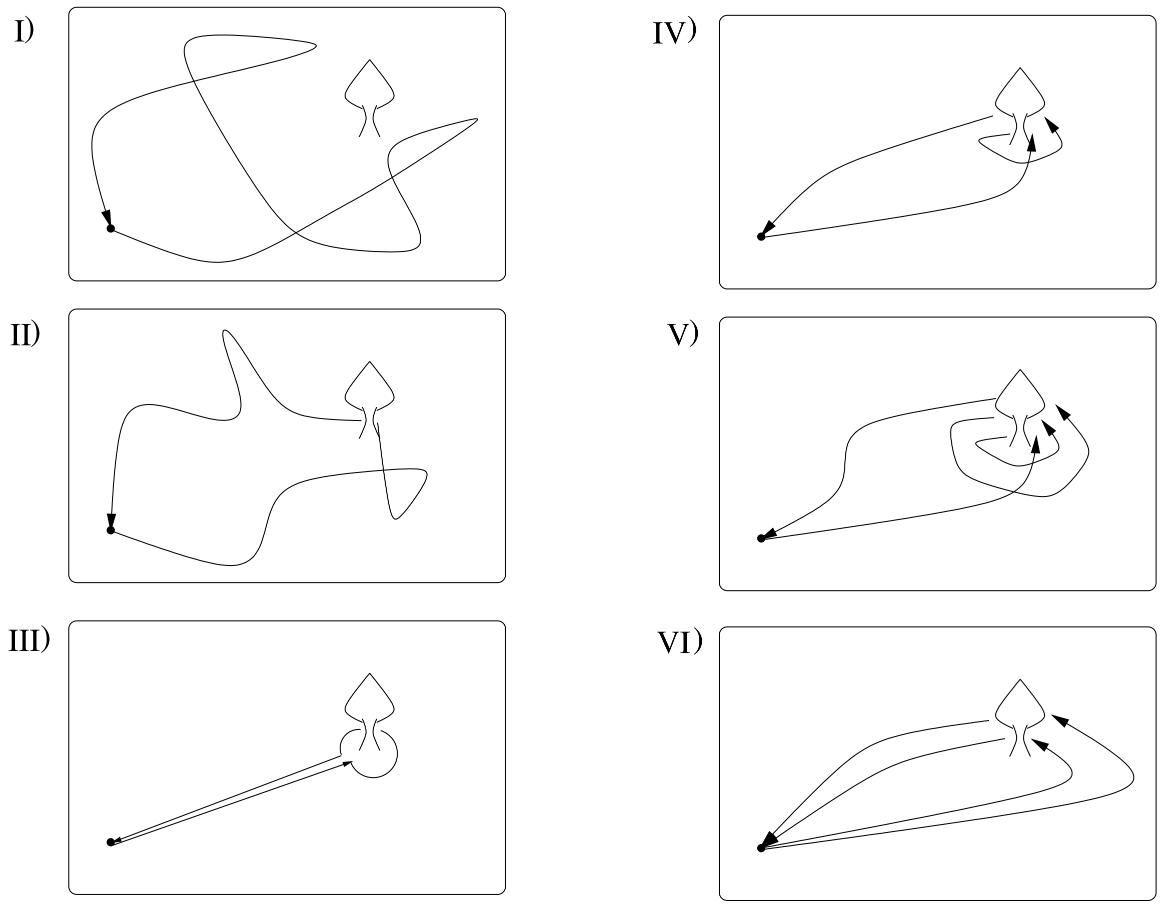

Figure 16: *Which of these paths are homotopic?*

9. What would happen if our camp was on the moon, and there was no fence surrounding it? In other words, we are considering a sphere with a tree. Will circling the tree be distinguishable from a point? Is there any path in this space that is not homotopic to P? What is the fundamental group of this space?

10. What is the fundamental group of (the surface of) a 3-dimensional cube? Is it different from the fundamental group of the sphere? How does it alter if we add a tree? What if we add another tree?

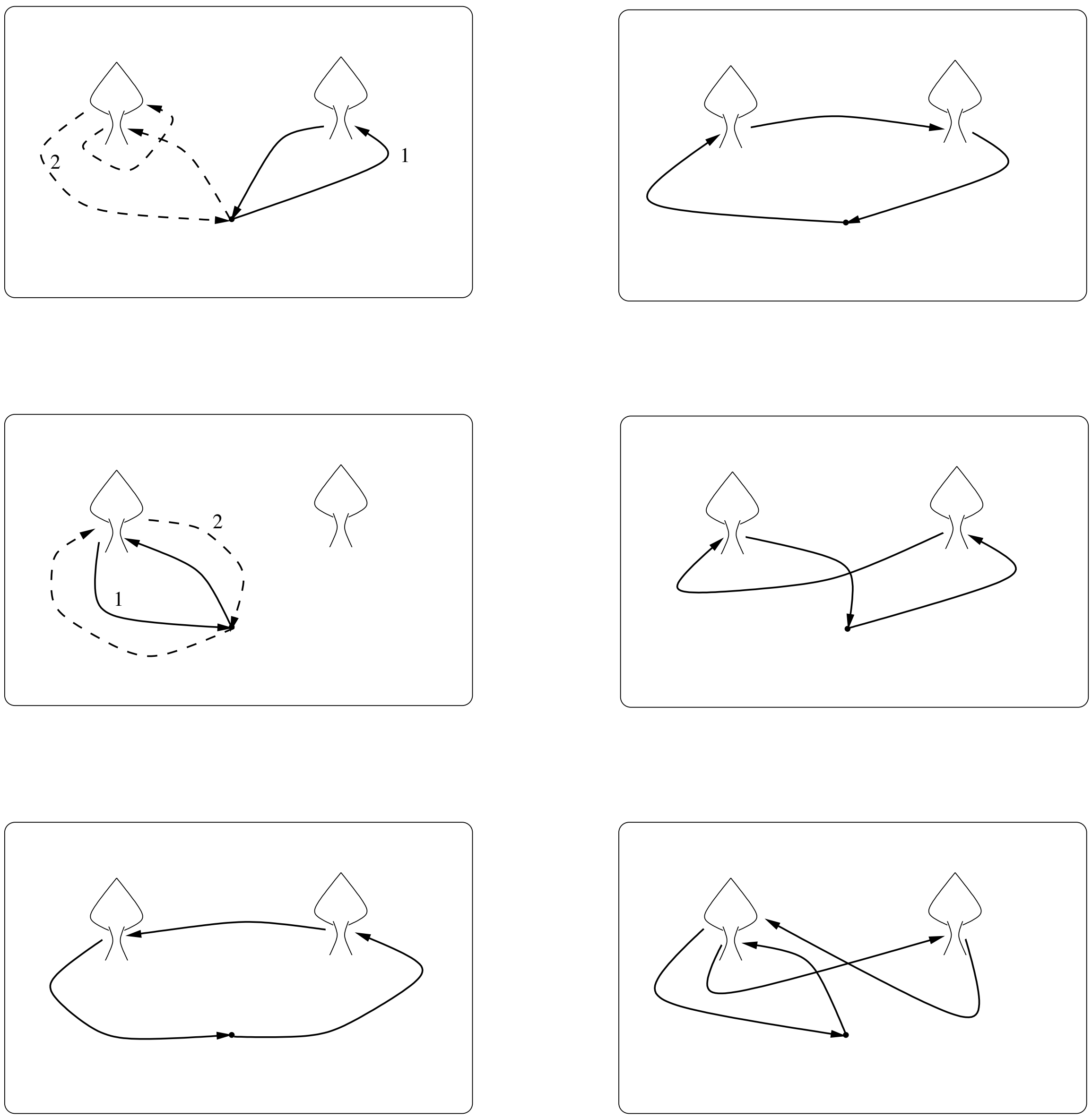

Figure 17: *Match a word in* a, b, a^{-1}, b^{-1} *to each path.*