Math Circle Contest

November 13th, 2002

Remember y'all that your little sisters (Nick and Dave) will be watching for the precision, conciseness, correctness and utter beauty of your answers!!!

(1) *JOY TO THE FISHES IN THE DEEP BLUE POND!*

(1a) Match each of the following species of fish with the appropriate evolution function;

(1b) Identify the equilibrium points on the graphs of the various evolution functions and distinguish between stable and unstable points.

Fish Limbaugh: is very conservative. It does not let any external condition change its population.

Tearful Tuna: is very emotional. They like to stay packed and cozy in the pond, but, if even only one individual passes away, all the others get depressed and soon the species becomes extinct.

Cooperative Cod: is very small, hence alone or in small groups is extremely vulnerable to predators. However, it is excellent at teamwork defense. If there's a sufficient number of fish in the pond, they will organize and their population will grow steadily.

Carp-e diem: is tenacious and optimistic. No matter how few they are to begin with, they'll find a way to survive and reproduce until they fill up the whole pond.

Barracuddler: is a glutton and, on a full stomach, a romantic. It will eat as much as it can and then go out on a date. As a result, when the food is abundant relative to the amount of fish, the population will dramatically increase. This will create a shortage of food that will cause the population size to shrink...and then grow, and then shrink...

N(t+1)

1

1   N(t)

N(t+1)

1

1   N(t)

N(t+1)

1

1   N(t)

N(t+1)

1

1   N(t)

N(t+1)

1

1   N(t)
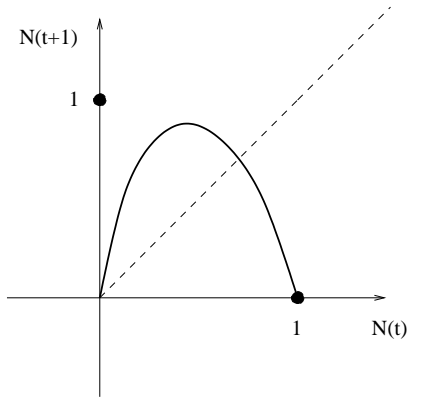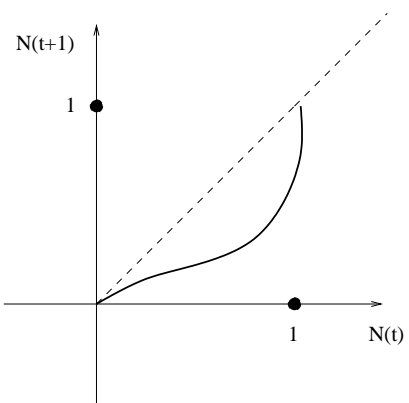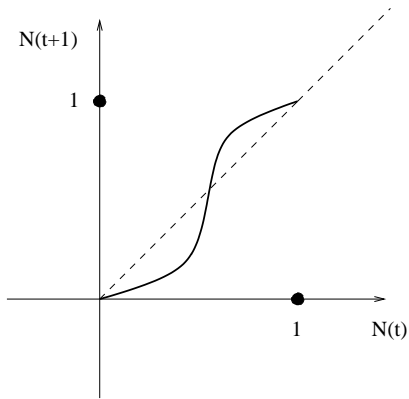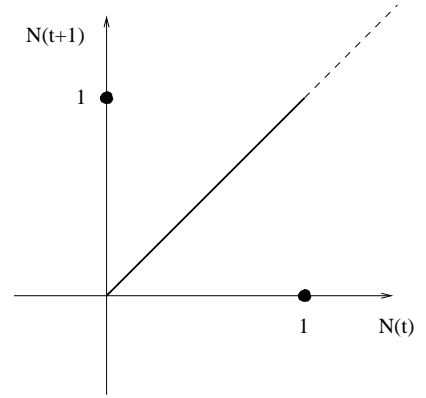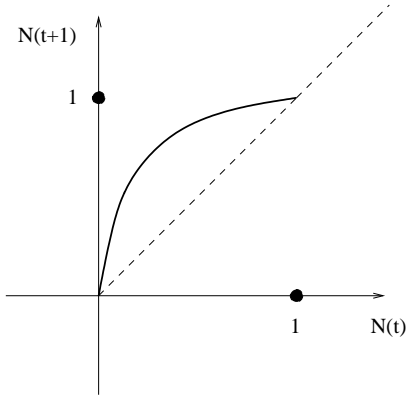
(2) STORE WARS

The town of Peacity always has exactly 10,000 residents. By statute, every time new residents move in or are born an equal number must move away or die.

It used to be that everyone in Peacity bought his or her food at Gracie's Grocery. But one day a branch of the national chain store Floormart opened in the neighboring town of Noville. Floormart is able to offer lower prices and more selection, but doesn't have the personal service and local products that characterize Gracie's.

Starting on opening week, and then for every subsequent one, 15% of the current Gracie's customers switch to Floormart because of the exceptional prices. However, it also turns out that each week 10% of the Peacity residents who were shopping at Floormart decide to return to the comforts of Gracie's, despite its higher prices. So for example, in the first week that Floormart is open, 1500 Peacity residents shop there and 8500 shop at Gracie's.

The dynamics of these store wars are represented by a directed graph, which shows how fractions shift between the two stores from week to week:
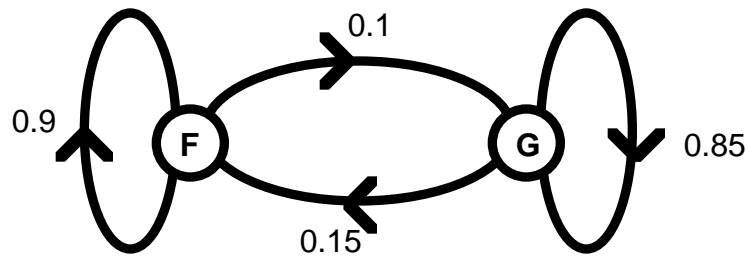
2a) How many Peacity customers shop at each store in the third week after Floormart opens?

2b) Imagine a slightly different scenario, in which Floormart heavily advertised its opening and attracted 5000 Peacity customers the first week (so that 5000 shopped at Gracie's). In this case how many customers will each store have in the third week, assuming the weekly transition dynamics remain the same?

2c) Assume the original scenario. Furthermore, Gracie's needs to average at least 3000 customers a week in order to stay in business. The manager notices that she is losing customers every week, and asks you to calculate how long she can hang on. What is your answer (and your explanation)? Would your answer change under the different scenario of (2b)?

(3) STORE WARS (continued)

The Gracie's manager hears that Floormart is going to try attracting more customers with heavy advertising, and that Floormart will also try to retain more customers by improving service. This could be big trouble for Gracie's and your friend the manager hires you to evaluate the situation.

You consider a general transition diagram assuming that Gracie's loses a fraction "b" of it's customers each week, and Floormart loses a fraction "a." (Both of these fractions are between 0 and 1, and can be partially influenced by advertising and customer service at each store.)
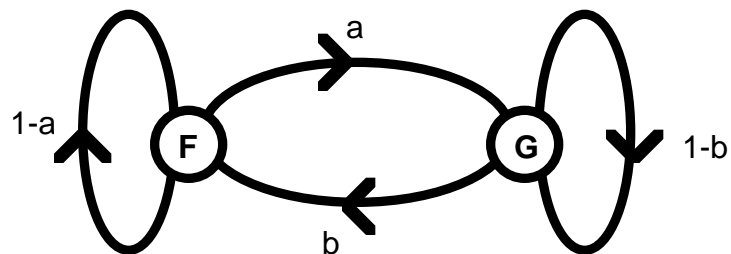
Although it is hard to be precise about how advertising and customer service affect the values of a and b, it is not too hard to answer the following purely mathematical question:

(3) What precise inequality involving a and b will guarantee that the steady state (long-term) number of customers who shop at Gracie's is at least 3000?

(4)

(Warning and disclaimer, for those who take us all too seriously. The following evolution function is not a meaningful model for any real biological situation. Or if it is, I didn't know about it. The scope of this problem is to test your grasp on the math that real biologists have to deal with in real life)

Consider the evolution function

$$N_{t+1} = \frac{N_t[(N_t - 4)^2 + (r - 4)^2]}{4}$$

depending upon the parameter r, which can vary among all possible positive numbers.

(4a) For a given value of r, what is the maximum number of equilibrium points that this function may have?

(4b) Can you spot some equilibrium point that is "easy to see"? (i.e. such that you don't need to go through any hard computation to find it).

(4c) Recall that the *bifurcation diagram* is the graph in the (N^*, r)-plane that plots the values of the equilibrium points for all the possible values of r. Draw the bifurcation diagram for this particular function and describe how the number of equilibrium points varies as the parameter r varies.