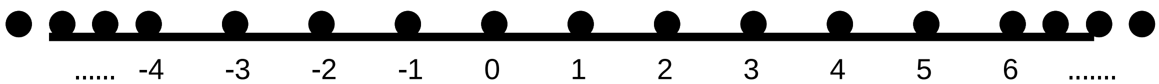
Math Circle Contest
October 9, 2002

Have fun and good luck! Make sure your answers pass the “little sister” test!

(1) LINEOLAND

In Lineoland there is an infinite train line, with stations numbered by the integers. A “leg” of a train journey consists of a trip between adjacent stations. The length of a trip is the number of legs you travel.



1a) Starting at station 0, how many trips of length 5 end at station 3?

1b) Starting at station 0, how many trips of length 101 end at station 3?

1c) Make a conjecture about the number of trips which start at station 0 and have length “n”, where n is any counting number. Your trips must start at station 0, but they may end anywhere. For example, there are four trips of length 2: one ending at station 2, one ending at station –2, and two ending at station 0.

1d) Prove your conjecture in (1c).

(2) SLOTHLAND

On his way back from Pentagonia, Abramo visits Slothland, inhabited by anthropomorphic sloths. Abramo wishes to introduce the card game of Poker to these 3-toed inhabitants. As you might expect, in sloth card games a “hand” consists of only 3 cards, although they do use a standard 52 card deck with 4 suits (hearts, diamonds, clubs, spades) and 13 denominations (2,3,4,5,6,7,8,9,10, J,Q,K,A).

Abramo has been dealing himself some practice hands and now has questions about how to rank the different possibilities. Here are the possible 3-card hands which Abramo is considering:

(i) A pair (exactly two cards of the same denomination, e.g. 2 Jacks.)
(ii) Three of a kind (e.g. 3 Jacks.)
(iii) Straight (3 cards in successive denominations but not all of the same suit, e.g. 9,10,J where one is a diamond and two are clubs.)
(iv) Flush (3 cards of the same suit, but not of successive denominations, e.g. 4, 7, 8 of hearts.)
(v) Straight-flush (A hand which is both a straight and a flush.)

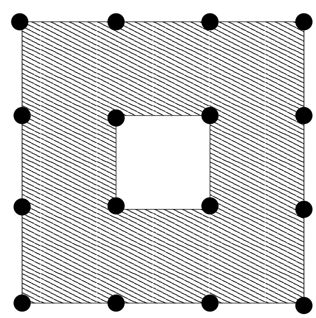
How should Abramo rank these 3-card hands so that less likely hands are more valuable?

(3) *HOLY PICK!*

A couple weeks ago we learned from Mr. Pick a really nifty way to compute the area of polygons inscribed in a square grid. It suffices to count the number of grid points that are inside the polygon (I) and the number of grid points that lay on the boundary (B). Then the area is given by the the formula:

$$A = \frac{B}{2} - 1 + I.$$

Unfortunately, Pick's formula doesn't work for the following shape:



$$8 = A \neq \frac{B}{2} - 1 + I = 7.$$

What goes wrong? The shape in question is NOT a polygon. It has a hole. As a matter of fact, Pick's formula can be modified so as to compute the area of shapes with holes in it (hence "Holy Pick"). We'll call $Pick^1$ the formula that allows us to compute the area of a shape with one hole given B and I.

3a) Figure out what is the correct formula for $Pick^1$.

3b) Prove your formula for $Pick^1$.

3c) Can you do the same for $Pick^n$, the formula that computes the area of shapes with n holes?