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Dan Ciubotaru: **Combinatorial geometry: points, lines, circles**

The following problems are taken from papers by Federico Ardila at San Francisco State University and Robert McCann at University of Toronto.

Warm-up problems:

Problem 1. Assume we have five arbitrary points in the plane, no three being collinear. Prove that there are four among them which form a convex quadrilateral.

Problem 2. Draw an arbitrary number of lines in the plane. Prove that the regions that are formed can be colored red and blue, so that any two regions which have a common segment are assigned different colors.

Counting the regions:

Problem 3. We draw n lines in the plane.

(a) Look at some examples $n = 1, 2, 3, 4$. What is the smallest number of regions and what is the largest number of regions that these lines can form?

(b) Assume we have n lines in general position. (This means that no three lines go through the same point, and no two lines are parallel.) Can we find a formula for the number of regions formed by these n lines?

Problem 4. Now let's look in three dimensions. Assume we have n planes in general position, that is, no two planes are parallel, no three planes go through the same line etc. How many regions do they form in the plane?

Problem 5. We have 20 circles passing through the origin, no two tangent at the origin, and also except for the origin, no three circles pass through a common point. How may regions in the plane are formed by these circles?

Halving lines and circles:

Problem 6. Assume we have $2n$ points in the plane, no three which are collinear. Show that there exists a halving line, i.e., a line which goes through two of the points and has $n - 1$ points on one side, and $n - 1$ points on the other side.

Problem 7. Assume we have $2n$ points as before. Prove that there are at least n different halving lines.

Problem 8. Let $2n+1$ points in the plane be given such that no three are collinear and no four are on the same circle. Prove that there exists a halving circle, i.e., a circle going through three of the points which has $n - 1$ points in its interior, and $n - 1$ points in its exterior.

Problem 9. Assume we have a set of $2n+1$ points as before. Show that for any two points in the set there exists at least one halving circle that goes through them. Conclude that there are at least $\frac{n(2n+1)}{3}$ halving circles. [1]

[1]F. Ardilla proved that, in fact, there are exactly n^2 halving circles for any $2n+1$ points in general position!