MATH CIRCLE SPRING CONTEST I

February 20, 2008

1.

The following four-by-four square consists of entries 0, 1, and 2 such that the row and column sums include each number from 1 to 8 exactly once:

$$\begin{array}{cccc} 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}$$

(a) Prove or disprove: there exist such a four-by-four square which has exactly three entries labeled 1. (Either find such a square or prove that no such square exists.)

(b) Prove or disprove: there exist such a four-by-four square which has exactly four entries labeled 1. (Either find such a square or prove that no such square exists.)

How many positive integers equal three times the sum of their digits? (For instance 27 is such a number since $27 = 3(2 + 7)$.)

Consider the following game played with points on the plane whose coordinates are both integers. A legal move consists of replacing a point (x, y) with a new point (x', y') assuming there is a third point (z, w) at the midpoint between (x, y) and (x', y'). Prove or disprove: there is a sequence of legal moves which carries the set of points $\{(0,0), (0,1), (1,0), (1,1)\}$ to the set $\{(0,0), (1,1), (2,-1), (2,0)\}$.

4.

Find all integers n which are equal to the square of the number of positive divisors of n. (For instance, nine is such a number: it has three positive divisors (1, 3, and 9), and three squared is nine.)