See-Saw Geometry and the Method of Mass-Points[1]

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Give me a place to stand on, and I can move the Earth.

— *Archimedes.*

1. Motivating Problems

Today we are going to explore a kind of geometry similar to regular Euclidean Geometry, involving points and lines and triangles, etc. The main difference that we will see is that we are going to give mass to the points in our geometry. We will call such a geometry, See-Saw Geometry (we'll understand why, shortly). Before going into the details of See-Saw Geometry, let's look at some problems that might be solved using this different geometry. Note that these problems are perfectly solvable using regular Euclidean Geometry, but we will find See-Saw Geometry to be very fast and effective.

Problem 1. Below is the triangle $\triangle ABC$. Side BC is divided by D in a ratio of $5 : 2$, and AB is divided by E in a ration of $3 : 4$. Find the ratios in which F divides the line segments AD and CE; *i.e.*, find $AF : FD$ and $CF : FE$. (Note: in Figure 1, below, only the *ratios* are shown; the actual lengths are unknown).

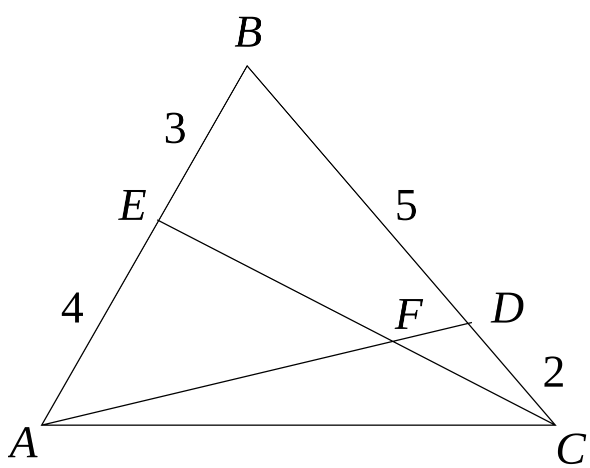

Figure 1

[1]My notes are shamelessly stolen from notes by Tom Rike, of the Berkeley Math Circle available at `http://mathcircle.berkeley.edu/archivedocs/2007_2008/lectures/0708lecturespdf/MassPointsBMC07.pdf` .

Problem 2. In Figure 2, below, D and E divide sides BC and AB, respectively, as before. Now, point G divides AC into a ratio of $3:7$. In what ratios does F divide the line segments ED and BG?

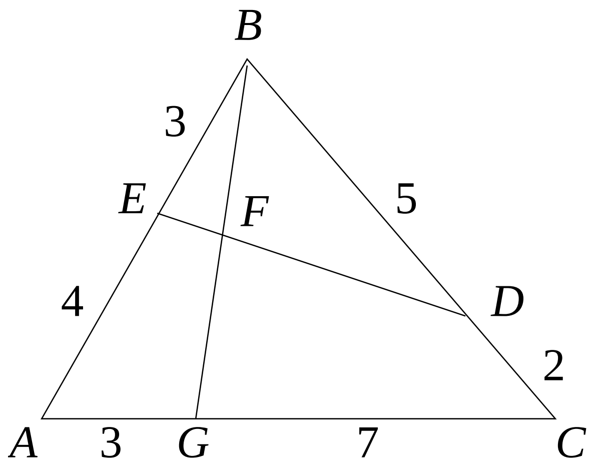

FIGURE 2

2. Definitions

We are now going to define See-Saw Geometry. In any kind of geometry–or any other branch of mathematics–we must first define what all the important characters are. In Euclidean Geometry, we defined what we mean when we say things like *point*, *line*, *circle*, or *triangle*. So we need to do the same for See-Saw Geometry. In See-Saw Geometry, the major character is the *mass point*.

Definition 1. A *mass point* is a pair (n, P), usually written nP, consisting of a positive real number n, the *mass*, and a point P in the underlying Euclidean space.

Definition 2. We say that two mass points are equivalent, and write $nP = mQ$ if $n = m$ and $P = Q$; *i.e.*, if they are located in the same point in space **and** have the same mass.

Definition 2. (Addition) $nA + mB = (n + m)F$ where F is on the line segment AB, and $AF : FB = m : n$.

Notice what the above definition says! If we take two mass points, nA and mB, we can add them! Their sum is another mass point, $(n + m)F$.

Question 1. Why is the name "See-Saw Geometry" appropriate? Hint: the point F was named so as to stand for *fulcrum*.

Definition 3.(Scalar Multiplication)] Given a mass point nP and a scalar $m > 0$, we define multiplication of a mass point by m as $m \cdot (nP) = (mn)P$.

3. Basic Properties in See-Saw Geometry

Property 1. (Closure). Addition of mass points produces a unique sum.
Property 2. (Commutivity). $nP + mQ = mQ + nP$.
Property 3. (Associativity). $nA + (mB + kC) = (nA + mB) + kC = nA + mB + kC$.
Property 4. (Idempotent). $nP + mP = (n + m)P$.
Property 5. (Distributivity). $k \cdot (nA + mB) = (kn)A + (km)B$.

Question 2. These five properties seem Very Important and Very Useful. At least they would be if they were true. Can you prove them?

Here is another property we might want to prove:

Property 6. (Subtraction) If $n > m$, then

$$nP = mQ + xX$$

may be solved for the unknown mass point xX. In particular, $xX = (n-m)R$ where R is on the line through PQ and $RP : PQ = m : (n-m)$. See Figure 3, below.

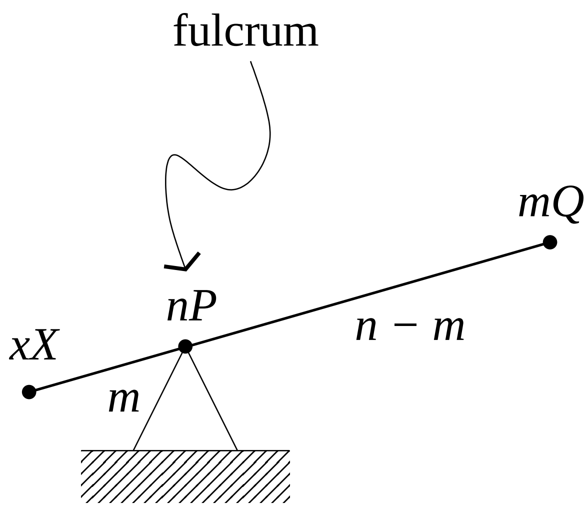

FIGURE 3. xX is the solution to $nP = mQ + xX$.

Question 2. Can you prove the rule of Subraction?

4. Problem Solving with See-Saw Geometry

Question 3. How can we use this idea of mass points to solve Problem 1? Try Problem 1 again, using See-Saw Geometry.

Question 4. Try Problem 2 again, using See-Saw Geometry. If you don't get this one now, that's okay; we will come back to it.

Problem 3. (Warm-up) If C is on the line AB, find x and $AC : CB$ so that both of the following conditions hold:

i. $3A + 4B = xC$;
ii. $7A + xB = 9C$.

Problem 4. In $\triangle ABC$, D is the midpoint of BC and E is the trisection point of AC nearer A (see Figure 4). Let G be the point of intersection between BE and AD. Find $AG : GD$ and $BG : GE$.

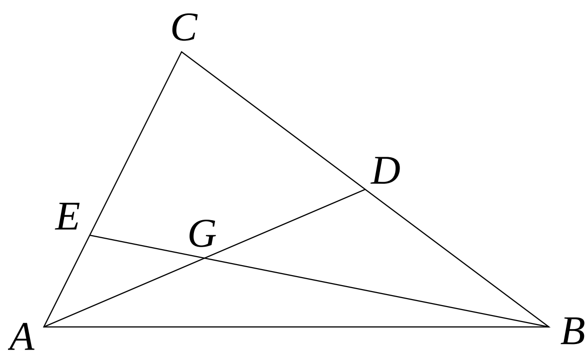

Figure 4

Problem 5. In $\triangle ABC$, D is on AB and E is on BC. Let F be the point of intersection between the line segments AE and CD. The following lengths are given: $AD = 3$, $DB = 2$, $BE = 3$ and $EC = 4$ (see Figure 5). Find the ratio $AF : FE$ in lowest terms.

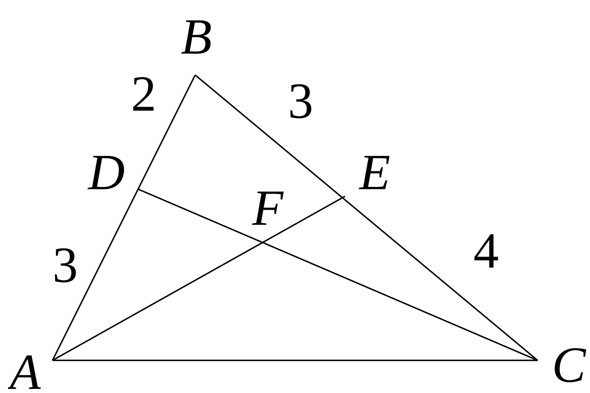

Figure 5

Problem 6. Show that the medians of a triangle are concurrent and the point of concurrency divides each median in a ratio of $2:1$.

5. More Problem Solving with See-Saw Geometry

We may take as given the following Problem Solving Techniques:

PST 1. Given two triangles with the same altitude, their areas are in the same ratio as their bases. In particular, if the triangles also have equal bases, then they have equal areas.

PST 2. A quadrilateral is a parallelogram if and only if its diagonals are mutual bisectors.

Now try the following problems:

Problem 7. Show that all six regions obtained by slicing a triangle via its three medians have the same area.

Problem 8. (Varignon's Theorem). If the midpoints of consecutive sides of a quadralateral are connected with line segments, the resulting quadralateral is a parallelogram.

Problem 9. In quadrilateral $ABCD$, the points E, F, G, and H are the trisection points of AB, BC, CD, and DA, respectively, nearer A, C, C, and A. Show that $EFGH$ is a parallelogram. (See Figure 6, below).

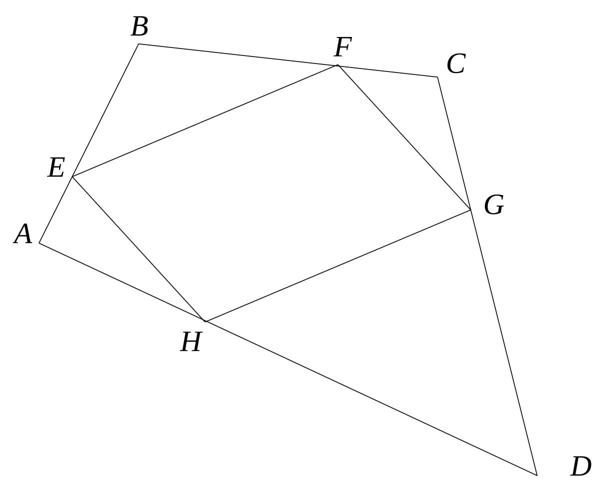

Figure 6

Problem 10. Generalize Problems 8 and 9 by considering points E, F, G, and H which divide the quadrillateral sides in corresponding ratios of $m : n$.

6. Angle Bisectors

We will take as given the following famous theorem:

Theorem 1. (Angle Bisector Theorem). An angle bisector in a triangle divides the opposite side in the same ratio as the other two sides. More precisely, in $\triangle ABC$, if ray $\vec{BD}$ bisects $\angle ABC$, then $AD : DC = AB : BC$.

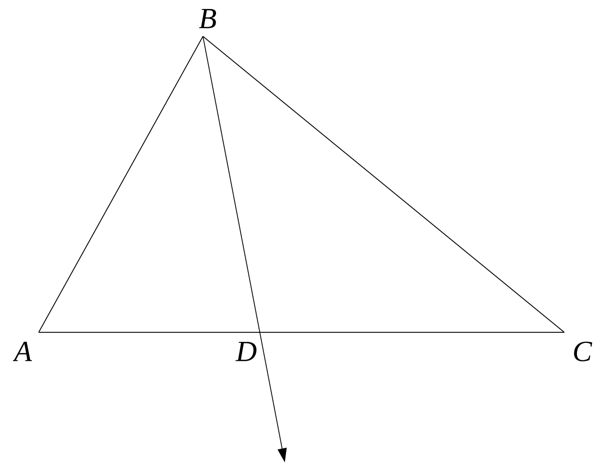

Figure 7

Problem 11. In $\triangle ABC$, let $AB = c$, $AC = b$, and $BC = a$. Assign a mass to each vertex equal to the length of the opposite side, resulting in mass points aA, bB, and cC. Show that the center of mass of this system is located on each angle bisector at a point corresponding to the mass point $(a + b + c)I$.

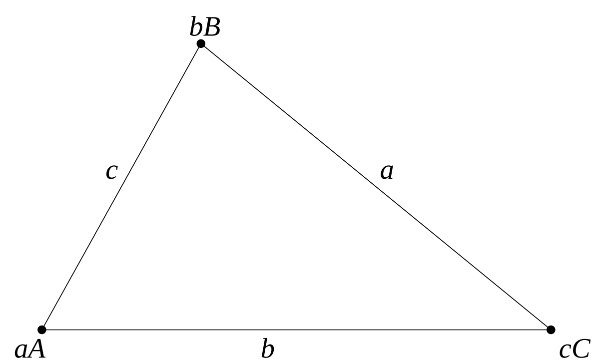

Figure 8

Problem 12. Use Problem 11 to prove that the angle bisectors of all three angles in a triangle are concurrent.

Many of you have probably seen the following from trigonometry:

Theorem 2. (Law of Sines). In $\triangle ABC$ where the opposite sides of $\angle A$, $\angle B$, and $\angle C$ are a, b, and c respectively, and R is the circumradius of $\triangle ABC$,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

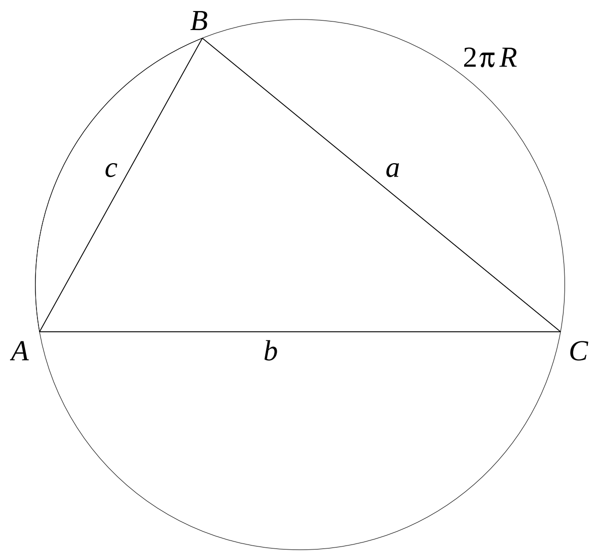

FIGURE 9

Problem 13. In $\triangle ABC$ with the bisector of $\angle B$ intersecting AC at D,

(1) Show that $AD : DC = \sin C : \sin A$;
(2) Let $\sin A = \frac{4}{5}$ and $\sin C = \frac{24}{25}$. The bisector BD intersects median AM at point E. find $AE : EM$ and $BE : ED$.

7. See-Saw Geometry and Area

Now we will strive to solve some area problems.

Problem 14. In $\triangle ABC$ D, E, and F are the trisection points of AB, BC, and AC nearer A, B, C, respectively (see Figure 10). Show that the area of $\triangle JKL$ is one-seventh the area of $\triangle ABC$. (*Hint*: First show $BJ : JF = 3 : 4$ and $AJ : JE = 6 : 1$, then use symmetry to conclude $DK : KL : LC = 1 : 3 : 3$)

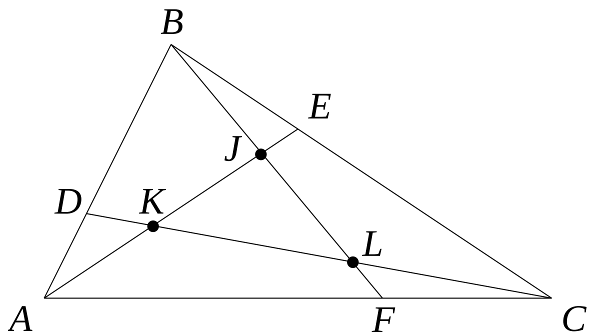

Figure 10

Problem 15. Generalize Problem 14 using points D, E, and F which divide the sides in a ratio of $1 : n$ (in place of $1 : 2$) to show that the ratio of the areas is $(n-1)^3 : (n^3-1)$.

8. Altitudes

Let BD be an altitude of length h of acute $\triangle ABC$ (See Figure 11). Note that $x \cdot \frac{y}{h} = y \cdot \frac{x}{h}$. So if we want D to be a balancing point of AC, we can choose the masses at A and C to be $\frac{y}{h}$ and $\frac{x}{h}$ respectively. Note that we could have chosen simply y and x; however, notice that $\frac{y}{h} = \cot A$ and $\frac{x}{h} = \cot C$. Therefore, chosing masses proportional to $\cot A$ and $\cot C$ will balance AC on the altitude BD.

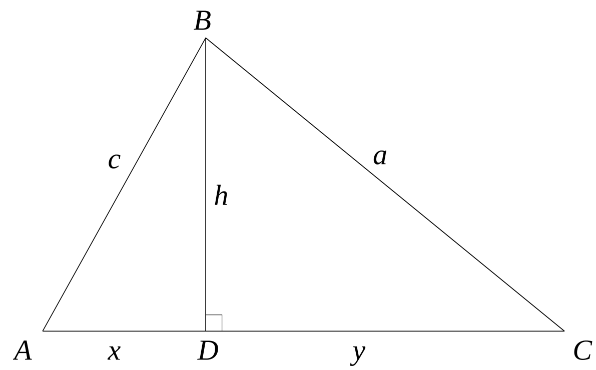

Figure 11

Problem 16. Let $\triangle ABC$ be a right triangle with sides $(8, 15, 17)$. (Note: $8^2 + 15^2 = 64 + 225 = 289 = 17^2$.) Let CD be the altitude of the hypotenuse and let the angle bisector of B intersect AC at F and CD at E. Show that

$$\begin{aligned} BE : EF &= 8 : 9, \\ CD : ED &= 17 : 8. \end{aligned}$$

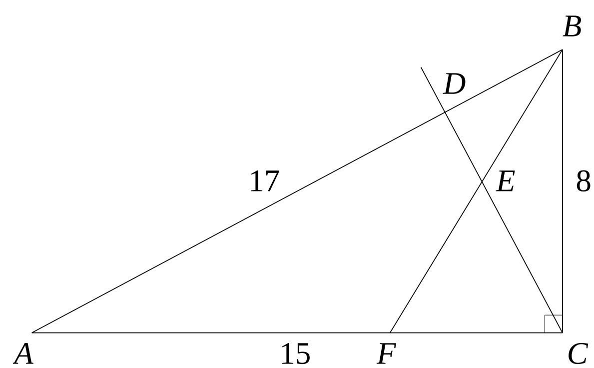

Figure 12

Problem 17. The sides of $\triangle ABC$ are $AB = 13, BC = 14$, and $AC = 15$. Let AD be an altitude of the triangle. The angle bisector of $\angle C$ intersects the altitude at E and AB at F. Find $CE : EF$ and $AE : ED$.

Problem 18. Prove that the altitudes of an acute triangle are concurrent using mass points.

9. Splitting Masses

Let's take another look at Problem 2.

Problem 2. In Figure 2, below, D and E divide sides BC and AB, respectively, as before. Now, point G divides AC into a ratio of $3:7$. In what ratios does F divide the line segments ED and BG?

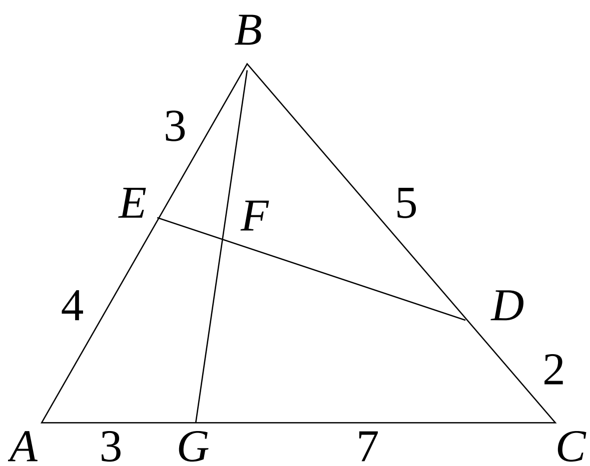

Figure 13

The key to solving this problem is to use the Idempotent Law:

Property 4. (Idempotent). $nP + mP = (n+m)P$.

Solution. Let's give a first approximation to the solution by assigning a mass of 4 to B and a mass of 3 to A in order to balance AB at E. This gives E a mass of 7.

Now balance AC at G. Thus, we need a mass of $9/7$ at C and a mass of $30/7$ at G.

Ah! But now it seems that we can't balance BC at D. Or can we? What if B were actually two points very close together, say B' and B'', so that $B'B''$ has a very small length. Then we could balance CB'' at D by giving B'' a mass of $18/35$, and D a mass of $9/5$.

In the language of our see-saw geometry, the mass point at B is really $4B + \frac{18}{35}B = \frac{158}{35}B$. Can we now use this to balance BG and ED at F? Try it!

Question 5. Why does this mass splitting work on transversals? Can you explain it in terms of see-saws?

Problem 19. In $\triangle ABC$, let E be on AB such that $AE : EB = 1 : 3$, let D be on BC such that $BD : DC = 2 : 5$, and let F be on ED such that $EF : FD = 3 : 4$. Finally, let ray $\vec{BF}$ intersect AC at point G. Find $AG : GC$ and $BF : FG$.

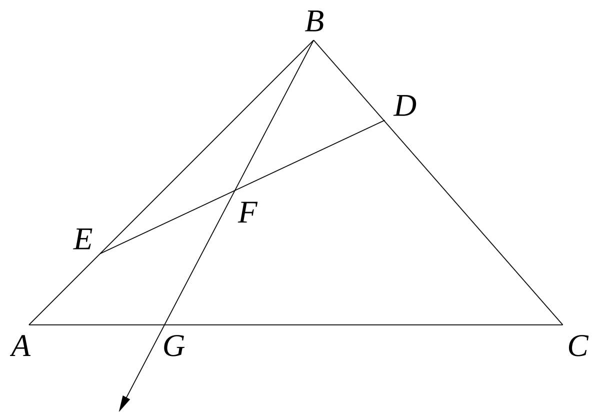

FIGURE 14

10. Contest Problems

(1) Point E is selected on side AB of $\triangle ABC$ in such a way that $AE : EB = 1 : 3$ and point D is selected on side BC so that $CD : DB = 1 : 2$. The point of intersection of AD and CE is F. Find
$$\frac{EF}{FC} + \frac{AF}{FD}.$$

(2) In $\triangle ABC$, C' is on side AB such that $AC' : C'B = 1 : 2$, and B' is on AC such that $AB' : B'C = 3 : 4$. If BB' and CC' intersect at P, and if A' is the intersection of the ray AP and BC, then find $AP : PA'$.

(3) In $\triangle ABC$, angle bisectors AD and BE intersect at P. If the sides of the triangle are $a = 3, b = 5, c = 7$, with $BP = x$ and $PE = y$, compute the ratio $x : y$ in lowest terms.

(4) In $\triangle ABC$, M is the midpoint of side BC, $AB = 12$ and $AC = 16$. Points E and F are taken on AC and AB respectively, and lines EF and AM intersect at G. If $AE = 2(AF)$ then find EG/GF.

(5) In $\triangle ABC$, points D and E are on AB and AC, respectively. The angle bisector of $\angle A$ intersects DE at F and BC at T. If $AD = 1, DB = 3, AE = 2$, and $EC = 4$, compute the ratio $AF : AT$.

(6) In $\triangle ABC$, $\angle B = 72°$, E is the midpoint of side AC and D is a point on side BC such that $2(BD) = DC$; AD and BE intersect at F. Find the ratio of the area of $\triangle BDF$ to the area of quadirlateral $FDCE$.

(7) In $\triangle ABC$, cevians AD, BE and CF intersect at a concurrent point P. The areas of $\triangle$'s PAF, PFB, PBD, and PCE are 40, 30, 35, and 84, respectively. Find the area of $\triangle ABC$.

(8) In $\triangle ABC$, D is on AB such that $AD : DB = 3 : 2$ and E is on BC such that $BE : EC = 3 : 2$. If ray DE and ray AC intersect at F, then find $DE : EF$.

(9) In a triangle, segments are drawn from one vertext the trisection points of the opposite side. A median drawn from a second vertex is divided, by these segments, in the continued ratio $x : y : z$. If $x \geq y \geq z$ then find $x : y : z$.

(10) In $\triangle ABC$, $\angle A = 45°$, and $\angle C = 30°$. If altitude BH intersects median AM at P, then $AP : PM = 1 : k$. Find k.

(11) The sides of a triangle are of lengths 13, 14, and 15. The altitudes of the triangle meet at point H. If AD is the altitude to the side of length 14, what is the ratio $HD : HA$?

(12) In $\triangle ABC$, A', B', and C' are on sides BC, AC, and AB, respectively. Given that AA', BB', and CC' are concurrent at the point O, and that
$$\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92,$$
find the value of
$$\left(\frac{AO}{OA'}\right) \cdot \left(\frac{BO}{OB'}\right) \cdot \left(\frac{CO}{OC'}\right).$$

(13) In $\triangle ABC$, if cevians AD, BE, and CF are concurrent at P, show that
$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

(14) Let P be an interior point of $\triangle ABC$ and extend lines from the vertices through P to the opposite sides. Let $AP = a$, $BP = b$, $CP = c$, and the extensions from P to the opposite sides all have length d. If $a + b + c = 43$ and $d = 3$ then find abc.

(15) Point P is inside $\triangle ABC$. Line segments are drawn from each vertex, through P, to points D, E, F on the sides opposite A, B, C, respectively. Given that $AP = 6, BP = 9, PD = 6, PE = 3$, and $CF = 20$, find the area of $\triangle ABC$.

(16) In $\triangle ABC$, let D and E be the trisection poins of BC with D closer to B and E closer to C. Let F be the midpoint of AC, and let G be the midpoint of AB. Let H be the intersection of EG and DF. Find the ratio $EH : HG$.