More Variations on a Fibonacci Theme[1]

(Notes for Week 2)

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Begin at the beginning and go on till you come to the end: then stop.

— Lewis Carroll

This week we start with some identities buried in Pascal's Triangle. Here is a warm-up:
Problem. Take a row of Pascal's Triangle and enumerate its (nonzero) entries as $P_0, P_1, P_2, \ldots$. Find a formula for

$$P_0 + P_2 + P_4 + \cdots$$

and prove that the formula is correct.

The problem follows pretty easily from considering the number of subsets of n objects which has an even number of elements.

Now that we're warmed up, here is a more challenging one.
Problem. Let a_i denote the number of odd entries in the ith row of Pascal's

Find (with proof) a formula for a_i. For instance, the zeroth row consists of the apex 1 and has a single odd element, so $a_0 = 1$. The first row, 1 1, has 2, so $a_1 = 2$. The second, 1 2 1, gives $a_2 = 2$, and the third, 1 3 3 1, gives $a_3 = 4$.

We can approach this problem much the way that we approached the tiling problems last week, that is by finding a natural recurrence which (if we're lucky, as we were last week with the Fibonacci recurrence) we can recognize as something familiar.

We start by writing out a few rows of Pascal's Triangle and starring the odd entries.

```
                        1*
                     1*    1*
                  1*    2     1*
               1*    3*    3*    1*
            1*    4     6     4     1*
         1*    5*    10    10    5*    1*
      1*    6     15*   20    15*   6     1*
   1*    7*    21*   35*   35*   21*   7*    1*
```

[1] An excellent reference for this material (and much more) is the book *Proofs That Really Count* by Arthur Benjamin and Jennifer Quinn published by the Mathematical Association of America. (At the time of writing, a significant portion of the book was freely available from `books.google.com` .) My notes are shamelessly stolen from notes by Hugo Rossi available at `http://www.math.utah.edu/teacherscircle/schedule.html` .

Now let's forget about the numbers are just look at the *'s:

```
              *
            *   *
          *       *
        *   *   *   *
      *               *
    *   *           *   *
  *       *       *       *
*   *   *   *   *   *   *   *
```

Now a pattern seems to be emerging: the starts keep reproducing themselves. For instance, if we break up the rows into chunks of four we get

```
              *
            *   *
          *       *
        *   *   *   *
-----------------------------
      *               *
    *   *           *   *
  *       *       *       *
*   *   *   *   *   *   *   *
-----------------------------
```

And it's clear that stars in rows 5–8 consist of two copies of the stars in rows 1–4. Actually this pattern appears everywhere. If we instead look at the first four rows broken into chunks of two, we get

```
      *
    *   *
-------------
  *       *
*   *   *   *
-------------
```

and again the stars in rows 3–4 consist of two copies of the stars in rows 1–2. We could even go back one step further to get

```
  *
-----
*   *
```

and the pattern once again holds: the stars in row 1 consists of two copies of the stars in row 0. So we are led to the following conjecture:

Conjecture. Write out the list of the numbers $a_0, a_1, \ldots$ of odd entries in the first 2^n rows of Pascal's Triangle. Then the list of the number of odd entries in the next 2^n rows of Pascal's triangle is *twice* the original list. More succinctly, for each $n \geq 1$ and $0 \leq j < 2^n$ the number of odd entries in rows $2^n + j$ is twice the number of odd entries in row j, i.e.

$$a_{2^n+j} = 2a_j.$$

Based on the pictures above the conjecture follows easily by induction and the following simple rules: odd plus odd is even, odd plus even is odd, and even plus even is even. I'll leave that to you.

Notice that the sequence whose jth term is the number of odd entries in the jth row of Pascal's Triangle is entirely determined by its zeroth value, namely 1, and the recurrence provided by the conjecture. The sequence starts with a 1, and then the next element must be twice it, namely 2. The next two elements must be double the preceding two, so we're up to

$$1, 2, 2, 4.$$

The next four entries must be double the first four, so we now have

$$1, 2, 2, 4, 2, 4, 4, 8,$$

and doubling again gives

$$1, 2, 2, 4, 2, 4, 4, 8, 2, 4, 4, 8, 4, 8, 8, 16, \ldots$$

and so on. Note that the elements in rows $2^n - 1$ of Pascal's Triangle are always all odd.

The question remains: does this sequence appear in a more familiar way? To answer it, we need only find another sequence with the same starting value and the same recurrence relation.

The appearance of all those 2^n's might suggest looking in the binary world. And, indeed, counting in binary satisfies a similar kind of recurrence. Suppose we write out the first 2^n n-digit binary numbers. The way to get the next 2^n binary numbers is to simply add a one in front of the list of n-digit numbers. For instance, if we start with the first 8 binary numbers (written as three digit numbers),

$$000, 001, 010, 011, 100, 101, 110, 111,$$

then we get the next 8 binary numbers by placing a 1 in front of the list we already have,

$$1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111.$$

Now, after padding with zeros, we have the list of the first 16 four digit binary numbers

$$0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111.$$

This is similar to the situation we encountered with Pascal's Triangle, but not precisely the same. We need to get doubling into the mix somehow.

Let's take a look instead at the sequence $b_0, b_1, \ldots$, where b_i is two raised to the power of the number of 1's in the binary representation of i. For instance, for the first 16 binary numbers, the list of the corresponding number of 1's is

$$0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, 4,$$

and after raising 2 to each of these powers we get

$$1, 2, 2, 4, 2, 4, 4, 8, 2, 4, 4, 8, 4, 8, 8, 16.$$

This is just the sequence we encountered by considering the odd entries in Pascal's Triangle. How can we *prove* they are the same? It is clear from the description of binary counting that the sequence $b_0, b_1, \ldots$ satisfies the same recurrence as indicated in the conjecture. Since $b_0 = 1$, we conclude $a_i = b_i$ for all i. We have thus proved:

Theorem. The number of odd entries in the ith row of Pascal's Triangle is 2^j where j is the number of 1's in the binary expression of i.

Here are some more problems for you to try your hand at.

Problem 3. Let f_n denote the number of tiling of the n-board by 1- and 2-tiles. Prove that
$$3f_n = f_{n+2} + f_{n-2}.$$

Problem 4. Show that if m divides n, then f_{m-1} divides f_{n-1}.

Problem 5. Let L_n denote the number of tilings of an n-bracelet by 1- and 2-tiles. Show
$$L_n = f_n + f_{n-2}.$$

Problem 6. Show that
$$L_n = L_{n-1} + L_{n-2}.$$

Problem 7. Prove that
$$f_{2n-1} = L_n f_{n-1}.$$

Problem 8. Prove that
$$5f_n = L_n + L_{n-2}.$$