Archimedes and the Arbelos[1]

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The mathematician's patterns, like the painter's or the poet's must be beautiful*; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.*

— *G.H. Hardy,* A Mathematician's Apology

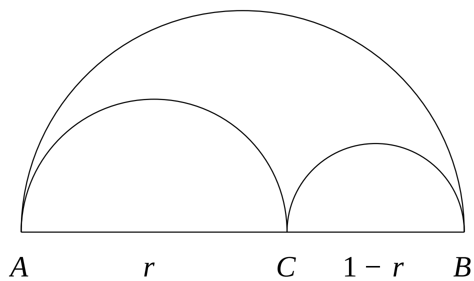


Figure 1. The Arbelos

Problem 1. We will warm up on an easy problem: Show that traveling from A to B along the big semicircle is the same distance as traveling from A to B by way of C along the two smaller semicircles.

Proof. The arc from A to C has length $\pi r/2$. The arc from C to B has length $\pi(1-r)/2$. The arc from A to B has length $\pi/2$. □

[1]My notes are shamelessly stolen from notes by Tom Rike, of the Berkeley Math Circle available at `http://mathcircle.berkeley.edu/BMC6/ps0506/ArbelosBMC.pdf` .

If we draw the line tangent to the two smaller semicircles, it must be perpendicular to AB. (Why?) We will let D be the point where this line intersects the largest of the semicircles; X and Y will indicate the points of intersection with the line segments AD and BD with the two smaller semicircles respectively (see Figure 2). Finally, let P be the point where XY and CD intersect.

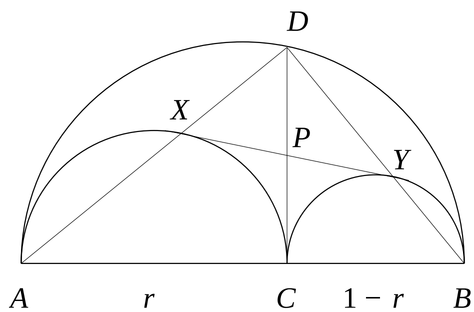


FIGURE 2

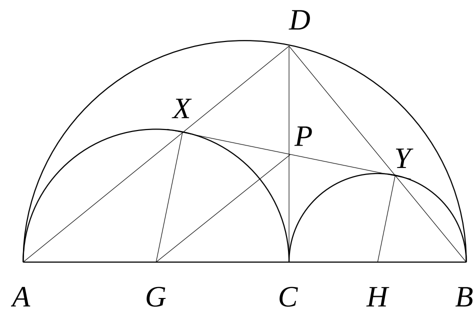
Problem 2. Now show that XY and CD are the same length, and that they bisect each other. What more can you say about these four points and their relationship with one another?

Proof. There is a famous theorem called Thales' Theorem: if A, B, and C are points on a circle, and AB forms a diameter of the circle, then $\measuredangle ABC$ is a right angle. This theorem implies that $\measuredangle AXC$, $\measuredangle ADB$, and $\measuredangle CYB$ are all right angles. Therefore, since three of the angles in the quadrilateral $XDYC$ are right, then it is a rectangle. Hence its diagonals are equal, $CD = XY$; and bisect each other.

Also note that X, D, Y, and C are all concircular (lie on the same circle). This is true for the vertices of any rectangle, but not true for any arbitrary four points in the plane. □

Problem 3. Show that XY is tangent to the small semicircles!

Proof. Let G be the center of the arc AXC and let H be the center of the arc CYB. Then $GX = GC$, and $XP = PC$ (by the previous problem). Therefore, $\measuredangle GXP = \measuredangle GCP = \pi/2$. Thus, XY is tangent at X. The case at Y is similar.



$\square$

Let MN be a perpendicular bisector of AB and a radius of the largest semicircle. Likewise, EG and FH are radii of the smaller semicircles, also perpendicular to AB. See Figure 3.

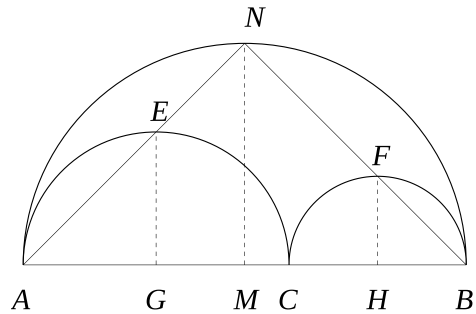


FIGURE 3

Problem 4. Show that A, E and N are colinear; as are B, F, and N. This is Proposition 1 in *The Book of Lemmas*.

Proof. $AG = GE$ so $\measuredangle GAE = \pi/4$. $AM = MN$ so $\measuredangle MAN = \pi/4$. Thus, AEN is a line segment.

The case for BFN is similar. $\square$

Problem 5. Show that the area enclosed by the three semicircles is the same as the area of the circle with diameter CD (see Figure 4). This is Proposition 4 in *The Book of Lemmas.*

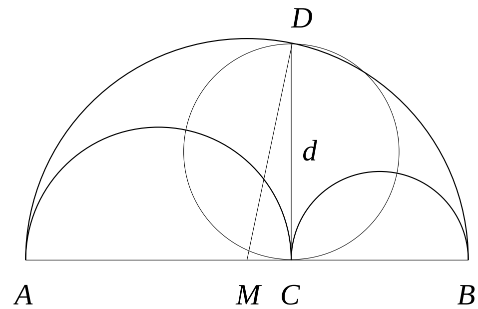


FIGURE 4

Proof. Let M be the center of the large semicircle. Let $d = CD$. It is easily seen through the Pythagorean Theorem applied to $\triangle MCD$ that $d^2 = r(1-r)$.

So the area of the circle is

$$\text{area}(\bigcirc) = \frac{\pi d^2}{4} = \frac{\pi r(1-r)}{4}.$$

The area of the Arbelos is easily checked by subtracting the areas of the small semicircles from the area of the large semicircle. We find the area is

$$\text{area}(\frown) = \frac{\pi}{8} - \frac{\pi r^2}{8} - \frac{\pi(1-r)^2}{8} = \frac{\pi r(1-r)}{4}.$$

The two areas agree. $\square$

Now we inscribe two circles on either side of the line segment CD, as in Figure 5. These circles are called the *Archimedean Twins*.

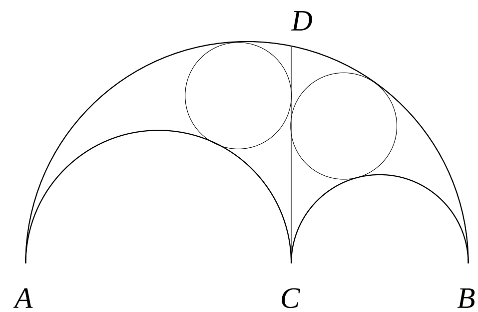


FIGURE 5

Problem 6. Show that the Archimedean Twins have equal diameters. Find this diameter in terms of r. (This is Proposition 5 in *The Book of Lemmas*.)

Problem 7. Construct (with proof), the Archimedean Twins in a given Arbelos using a straightedge and compass (*i.e.*, Euclidean construction).

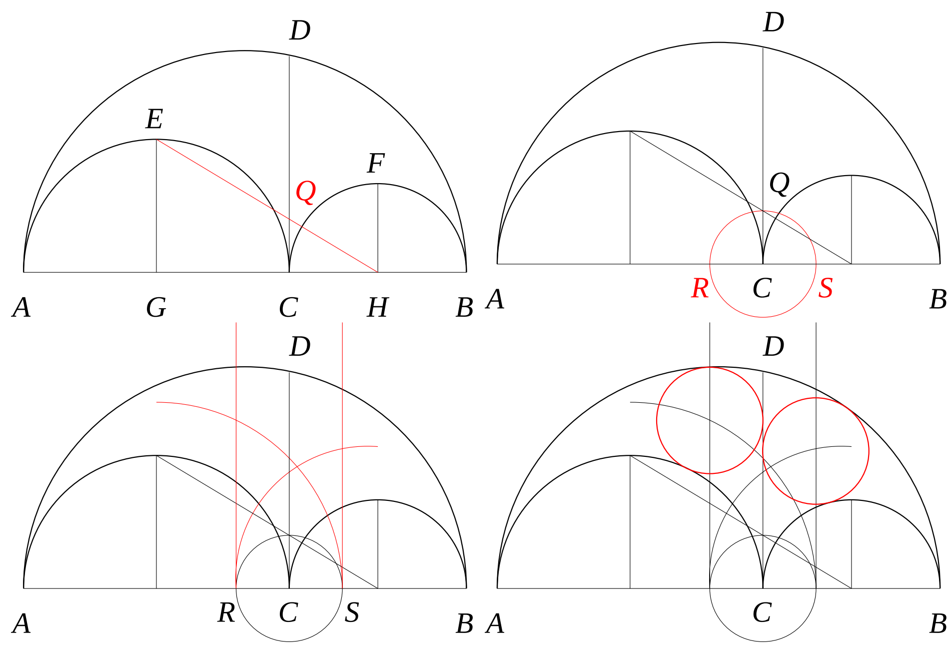


FIGURE 6. Euclidean Construction of the Archimedean Twins.

Proof. Construct GE and HF as perpendicular bisectors to AC and CB, respectively. Construct the line EH, and let Q be the intersection of EH and CD. Then it is easily shown via similar triangles that $CQ = r(1-r)/2$, which is the radius of the Twins. Construct the arc RQS, then the lines perpendicular to AB at R and S. Then construct the arcs centered at A and B with respective radii AS and BR. The intersections of the perpendiculars through R and S with these arcs are the centers of the Twins. □

Problem 8. Find the diameter of the circle tangent to all three semicircles that form the Arbelos (see Figure 7) in terms of r. This is Proposition 6 of *The Book of Lemmas*.

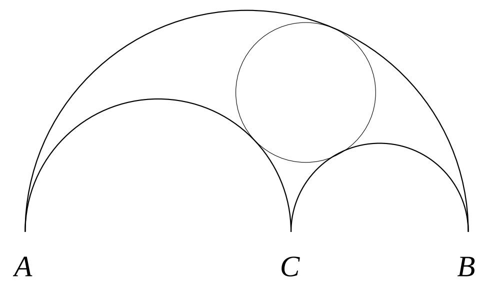


FIGURE 7

We can expand on this idea of inscribing circles in the Arbelos. Let C_1 be the circle given in the previous problem. Then let C_n represent the nth circle in the chain as in Figure 8.

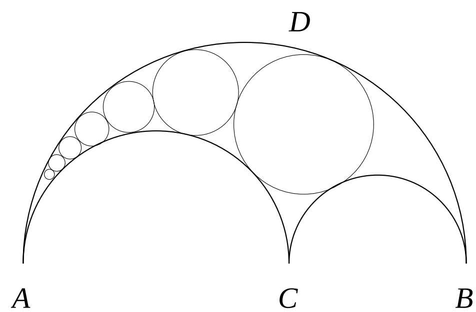


FIGURE 8

Problem 9 (Pappus). Show that the distance from the segment AB to the center of the nth circle, C_n in the chain is exactly $n \cdot d_n$, where d_n is the diameter of C_n. (Hint: if you know how to use inversion in a circle orthogonal to C_n, this is much easier. However, Pappus orginially solved this in a much different way.)

Problem 10. Show that the centers of the circles in the above chain all lie on an ellipse with foci at the centers of the two semicircles in which the chain is inscribed. In fact, any circle so inscribed between two such semicircles will have its center on this ellipse.

Now let's turn our attention back to the Archimedean Twins. We will see that they occur a lot within and around the Arbelos (in fact, there are infinitely many such occurances of the Twins!).

Our first recurrance of the Twins is simple enough: Inside each of the two smaller semicircles of the Arbelos, construct a similar Arbelos to the original (see Figure 9).

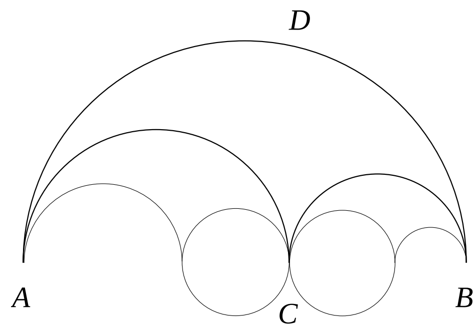


FIGURE 9

Problem 11. Show that the two middle semicircles of these new Arbelosi are actually Archimedean Twins.

Let S and T be the points where the first circle in our Pappus chain is tangent to each of the smaller semicircles in the Arbelos (see Figure 10).

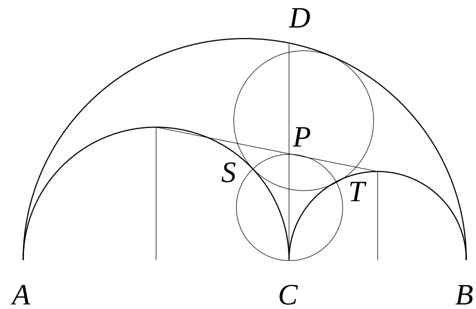


FIGURE 10

Problem 12. Show that the circle formed by S, T, and C is an Archimedean Twin.

Problem 13. Moreover, show that this circle passes through the point P, where P is the intersection of CD with the line connecting the zeniths of the small semicircles (see Figure 10).

Consider the circles with centers A and B, and whose respective radii are AC and BC. Construct a circle tangent to each of these, and inscribed in the large semicircle of the Arbelos (see Figure 11).

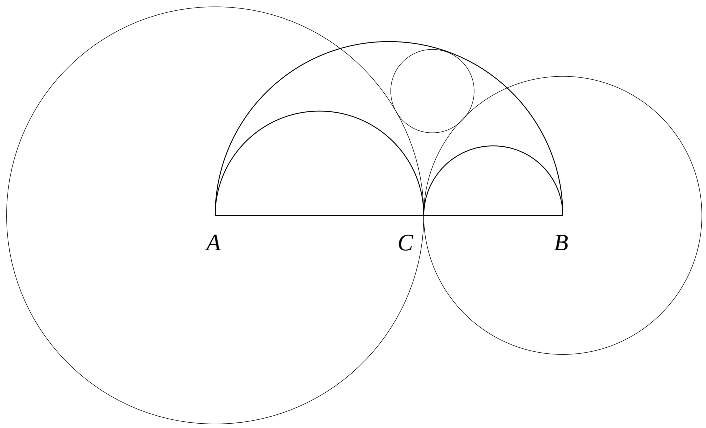


FIGURE 11

Problem 14. Show that this circle is an Archimedean Twin.

Let's generalize the previous result. First, draw the line through the center of this most recent occurance of a Twin, perpendicular to the baseline AB. This line is known as the Schoch line, and will allow us to find an infinite family of Archimedean Twins, known as the Woo circles.

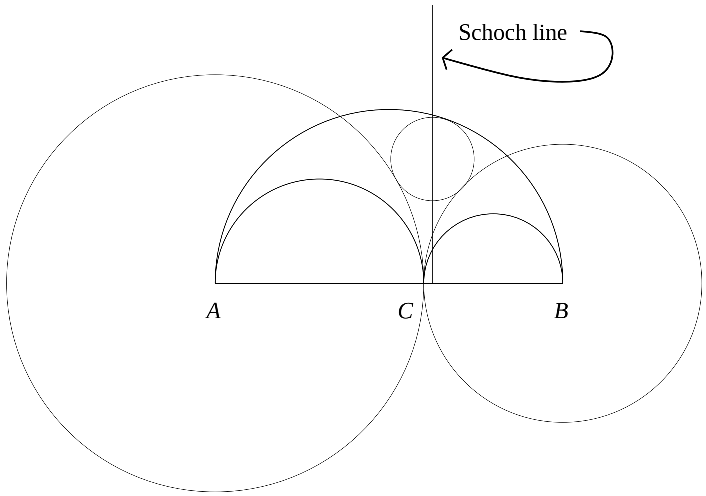


FIGURE 12

Let m be any positive number. Construct two circles so their centers lie on the baseline of the Arbelos, and so their respective radii are mr and $m(1-r)$. Next consider the circle which is tangent to each of these, and whose center lies on the Schoch line. See Figure 13

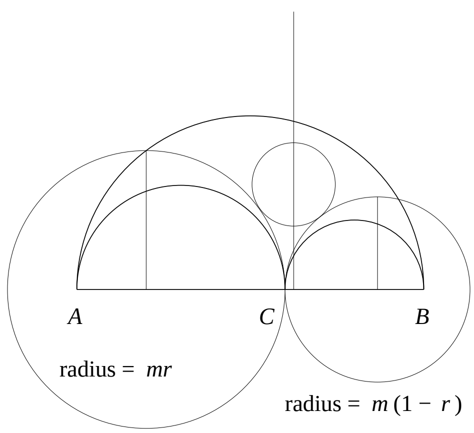


FIGURE 13

Problem 15. Show that this circle is also an Archimedean Twin (thus, we have an infinite family of Twins–one for each value of m).