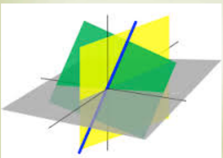
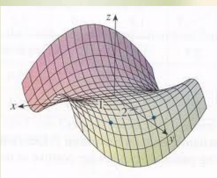
The Cross Product



$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

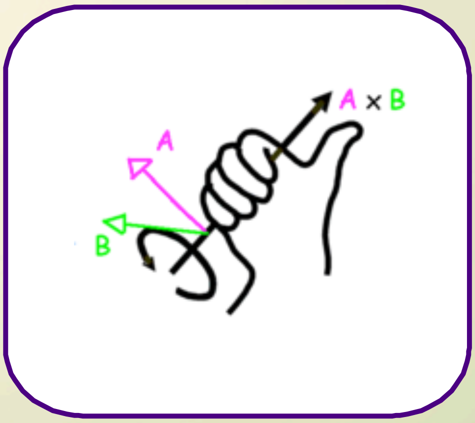
$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$



The Cross Product

$\vec{u} \times \vec{v}$ read "u cross v"

$$\vec{u}\times\vec{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$$

where $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$$

$$= \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$$

EX 1 If $\vec{a} = \langle 3,3,1 \rangle$ and $\vec{b} = \langle -2,-1,0 \rangle$ and $\vec{c} = \langle -2,-3,-1 \rangle$,

find $\vec{a} \times (\vec{b} \times \vec{c})$.

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{a} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & 0 \\ -2 & -3 & -1 \end{vmatrix}$$

$$= \vec{a} \times \left[(1-0)\hat{i} - (2-0)\hat{j} + (6-2)\hat{k} \right]$$

$$= \vec{a} \times \langle 1,-2,4 \rangle$$

$$= \langle 3,3,1 \rangle \times \langle 1,-2,4 \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 3 & 1 \\ 1 & -2 & 4 \end{vmatrix} = \hat{i}(12+2) - \hat{j}(12-1) + \hat{k}(-6-3)$$

$$= 14\hat{i} - 11\hat{j} - 9\hat{k}$$

or $\langle 14,-11,-9 \rangle$

Note: cross product of 2 vectors returns a <u>vector</u>!!! (dot product returns a scalar)

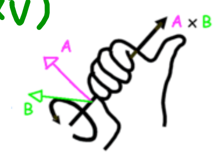
Theorem A

Let $\vec{u}$ and $\vec{v}$ be 3-D vectors and θ is the angle between them.

Then

1) $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0 = \vec{v} \cdot (\vec{u} \times \vec{v})$ $\Rightarrow \vec{u} \perp (\vec{u} \times \vec{v})$
$\vec{v} \perp (\vec{u} \times \vec{v})$

2) $\vec{u}, \vec{v}$ and $\vec{u} \times \vec{v}$ form a right-handed triple.



3) $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$

Note: for $\|\vec{u} \times \vec{v}\| = 0$ to be true,
then $\sin\theta = 0$ must be true
$\Rightarrow \theta = 0$ (or π)
$\Rightarrow \vec{u}$ and $\vec{v}$ are pointing in same dir.
(or opposite directns)

Theorem B

Two 3-D vectors, $\vec{u}$ and $\vec{v}$ are parallel iff $\vec{u}\times\vec{v}=0$.

because $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$

if $\vec{u}\parallel\vec{v}$, then $\theta=0 \Rightarrow \sin 0 = 0$

$\rightarrow \|\vec{u}\times\vec{v}\|=0$

$\Rightarrow \vec{u}\times\vec{v}=0$

EX 2 Find the plane through these points.

$P_1(-1,3,0)$, $P_2(5,1,2)$ and $P_3(4,-3,-1)$

eqn of plane: $\langle A,B,C\rangle\cdot\langle x-x_1, y-y_1, z-z_1\rangle=0$

$\langle A,B,C\rangle=\vec{n}$ to plane, (x_1,y_1,z_1) pt on plane

a vector that's $\perp$ to the plane is $\perp$ to every vector on that plane.

$\vec{u}=\overrightarrow{P_1P_2}=\langle 5-(-1), 1-3, 2-0\rangle=\langle 6,-2,2\rangle=2\langle 3,-1,1\rangle$

$\vec{v}=\overrightarrow{P_2P_3}=\langle 4-5,-3-1,-1-2\rangle=\langle -1,-4,-3\rangle=-\langle 1,4,3\rangle$

$$\vec{n}=\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&-1&1\\1&4&3\end{vmatrix}=\hat{i}(-3-4)-\hat{j}(9-1)+\hat{k}(12+1)$$

$=\langle -7,-8,13\rangle$

$P_1(-1,3,0)$

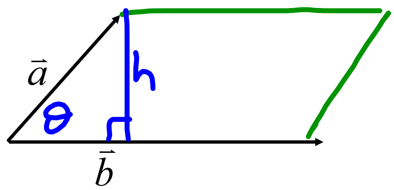
$\langle -7,-8,13\rangle\cdot\langle x+1, y-3, z\rangle=0$

$-7(x+1)-8(y-3)+13z=0$

$-7x-8y+13z=-17$

$\boxed{7x+8y-13z=17}$

EX 3 Find the area of a parallelogram with vectors $\vec{a}$ and $\vec{b}$ as adjacent sides.



$A = \text{base} * \text{ht}$

$A = \|\vec{b}\|\, h$

notice: ② $\sin\theta = \dfrac{h}{\|\vec{a}\|}$

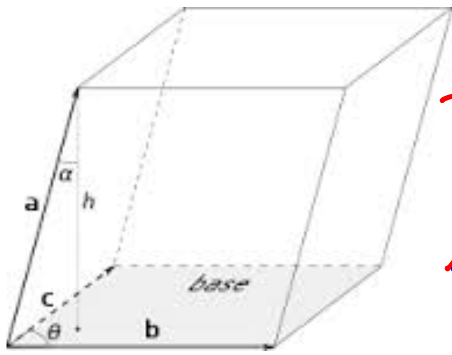
also ① $\sin\theta = \dfrac{\|\vec{a}\times\vec{b}\|}{\|\vec{a}\|\,\|\vec{b}\|}$

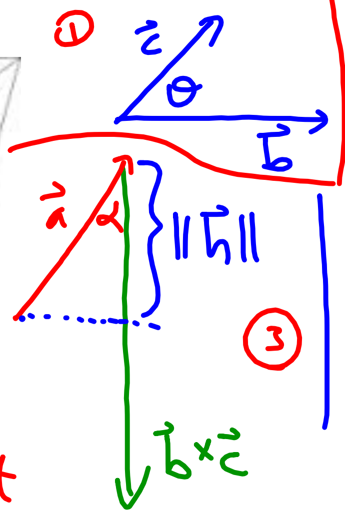
$\Rightarrow \quad \dfrac{h}{\|\vec{a}\|} = \dfrac{\|\vec{a}\times\vec{b}\|}{\|\vec{a}\|\,\|\vec{b}\|} \iff h = \dfrac{\|\vec{a}\times\vec{b}\|}{\|\vec{b}\|}$

$\Rightarrow A = \cancel{\|\vec{b}\|}\, \dfrac{\|\vec{a}\times\vec{b}\|}{\cancel{\|\vec{b}\|}} = \|\vec{a}\times\vec{b}\|$

$\boxed{A = \|\vec{a}\times\vec{b}\|}$

EX 4 Find the volume of a parallelogram prism(box) determined by the sides $\vec{a}$, $\vec{b}$ and $\vec{c}$.





② direction of $\vec{h}$ = dir. of normal vector of base plane, i.e. direction of $\vec{b}\times\vec{c}$

③ $\|\vec{h}\|$ = projectn of $\vec{a}$ onto $\vec{b}\times\vec{c}$

V = area of base * ht

$V = \|\vec{b}\times\vec{c}\| * ht = \|\vec{b}\times\vec{c}\|\,\|\vec{h}\|$

$\|\vec{h}\| = \|pr_{\vec{b}\times\vec{c}}\vec{a}\|$

$$pr_{\vec{b}\times\vec{c}}\vec{a} = \frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\|\vec{b}\times\vec{c}\|}\left(\frac{\vec{b}\times\vec{c}}{\|\vec{b}\times\vec{c}\|}\right) \Rightarrow \|pr_{\vec{b}\times\vec{c}}\vec{a}\| = \frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\|\vec{b}\times\vec{c}\|}$$

$$\Rightarrow V = \cancel{\|\vec{b}\times\vec{c}\|}\,\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\cancel{\|\vec{b}\times\vec{c}\|}}$$

$$\boxed{V = \vec{a}\cdot(\vec{b}\times\vec{c})}$$

volume for parallelopiped (aka parallelogram prism)

Theorem C Properties of Cross Product

$\vec{u}$, $\vec{v}$ and $\vec{w}$ are 3-D vectors and $k \in \Re$:

1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

2) $u \times (v + w) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$ (left distributive property)

3) $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$

4) $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$ and $\vec{u} \times \vec{u} = \vec{0}$

5) $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$

6) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

$$\hat{i} \times \hat{j} = \hat{k} \qquad \hat{j} \times \hat{k} = \hat{i} \qquad \hat{k} \times \hat{i} = \hat{j}$$

EX 5 State whether each of the following expressions make sense or not. If it makes sense, tell if the result is a scalar or a vector.

✓ a) $\vec{u}\cdot(\vec{v}\times\vec{w})$ vector dot vector returns scalar

✓ b) $\vec{u}+(\vec{v}\times\vec{w})$ vector added to another vector ⇒ returns vector

✗ c) $(\vec{a}\cdot\vec{b})\times\vec{c}$ Scalar crossed w/ a vector: cannot do

✗ d) $(\vec{a}\times\vec{b}+k)$ vector plus scalar: cannot do

✓ e) $(\vec{a}+\vec{b})\times(\vec{c}+\vec{d})$ vector cross vector ⇒ returns vector

✓ f) $(\vec{a}\cdot\vec{b}+k)$ Scalar + scalar ⇒ returns scalar

EX 6 Find the equation of a plane through $(5,-1,2)$ that is perpendicular to the line of intersection of the planes

① $4x-3y+2z=1$ and ② $2x-y+z=11$.

$\vec{n} = \langle A, B, C \rangle$ normal vector

eqn of plane: $\langle A, B, C \rangle \cdot \langle x-5, y+1, z-2 \rangle = 0$

notice: the cross product of plane ① normal vector w/ plane ② normal vector returns a vector that lies on both planes ① and ② $\Rightarrow$ this is the line of intersection

plane ① normal: $\langle 4, -3, 2 \rangle$

plane ② normal: $\langle 2, -1, 1 \rangle$

$$\vec{n} = \langle 4,-3,2 \rangle \times \langle 2,-1,1 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -3 & 2 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= \hat{i}(-3+2) - \hat{j}(4-4) + \hat{k}(-4--6)$$

$$= \langle -1, 0, 2 \rangle$$

$$\langle -1, 0, 2 \rangle \cdot \langle x-5, y+1, z-2 \rangle = 0$$

$$-(x-5) + 2(z-2) = 0$$

$$-x + 2z = -1$$

$$\boxed{x - 2z = 1}$$