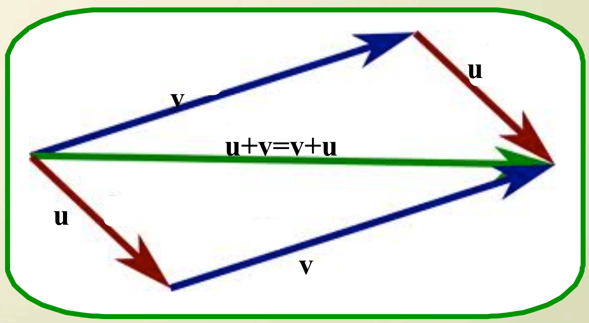
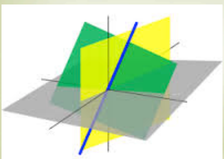
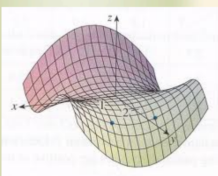
A Geometric and Algebraic Approach to Vectors





$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

VECTORS (Geometric Approach)

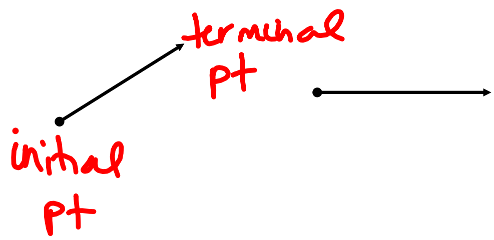
Scalar : a $\mathbb{R}$ number constant

Vector : directed line segment

① Magnitude : length

② Direction : angle from a reference direction

note: location does not matter for a vector



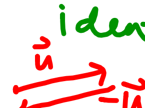
$\vec{u} = \vec{v}$ if they have the same magnitude and direction.

(note: vector u will be written as $\vec{u}$ or as boldfaced u) (or $\underline{u}$)

zero vector $\Rightarrow$ $\vec{0}$ and $\vec{0} + \vec{u} = \vec{u} + \vec{0} = \vec{u}$

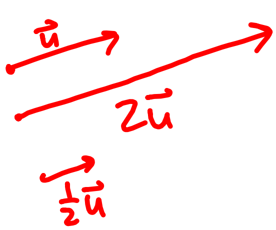
a vector w/ zero length; the additive identity

$-\vec{u}$ $\Rightarrow$ $\vec{u}$ vector pointing in the opposite direction

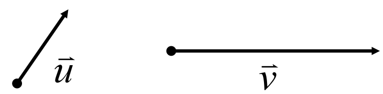


scalar multiple of $\vec{u} \Rightarrow c\vec{u}$, where c is a real number,

means we have a vector in the direction of $\vec{u}$ but scaled in length.

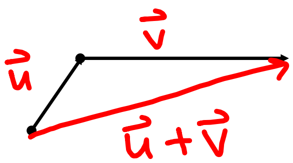


Adding vectors $\Rightarrow$ $\vec{u}+\vec{v}$



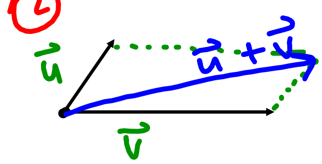
(Geometric)

①



place $\vec{u}$ & $\vec{v}$ "tail to head"
then connect initial pt to final terminal pt

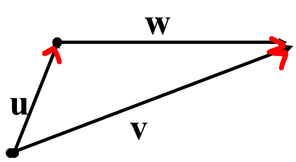
②



- place $\vec{u}$ & $\vec{v}$ with coincident initial pts
- create parallelogram
- $\vec{u}+\vec{v}$ is the vector from initial pt to opp. vertex of parallelogram (along diagonal)

EX 1

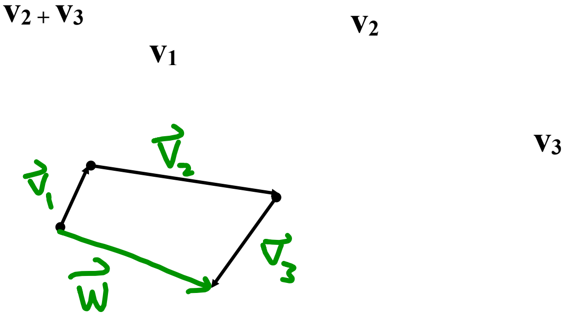
Express **w** in terms of **u** and **v**.



$\vec{u} + \vec{w} = \vec{v}$

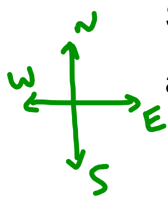
EX 2

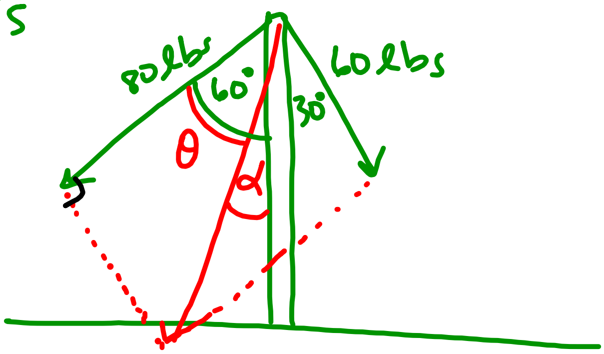
Draw **w** where $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$



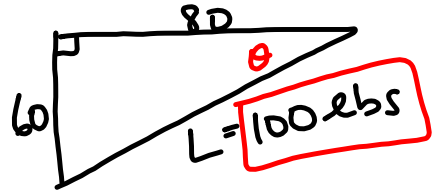
EX 3

Mark pushes on a post in the direction S 30° E with a force of 60 lbs. Dan pushes on the same post in the direction S 60° W with a force of 80 lbs. What are the magnitude and direction of the resulting force?





we have a rt. Δ.



$$\theta = \arctan\left(\frac{60}{80}\right) = \arctan\left(\frac{3}{4}\right)$$

$$60° = \theta + \alpha$$

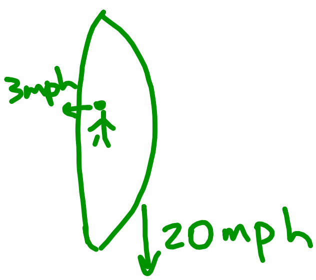
$$\alpha = 60° - \theta = 60° - \arctan\left(\frac{3}{4}\right)$$

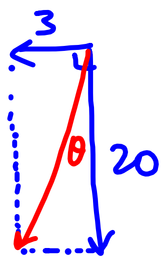
$$\boxed{\alpha \simeq 23.1°}$$

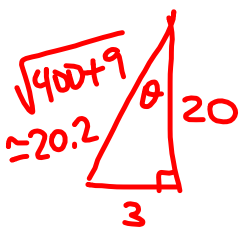
resulting force has magnitude of 100 lbs w/ direction S 23.1° W

EX 4

A ship is sailing due south at 20 mph. A man walks west across the deck at 3 mph. What are the magnitude and direction of his velocity relative to the surface of the water?







$\theta = \arctan\left(\frac{3}{20}\right)$

$\simeq 8.5°$

(note: we are assuming that surface of water is still)

resulting velocity:

$\boxed{20.2 \text{ mph}, \quad S\ 8.5° W}$

Vectors (Algebraic Approach)

If we place our vector on a Cartesian Coordinate system with its tail at the origin, then its head will end at some point $\underbrace{(u_1, u_2, u_3)}_{\text{pt}}$. We say that $\mathbf{u} = \underbrace{\langle u_1, u_2, u_3 \rangle}_{\text{vector}}$

u_1, u_2 and u_3 are called components of $\boldsymbol{u}$.

(iff = if and only if)

$\mathbf{u} = \mathbf{v}$ iff $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$

$\mathbf{u} + \mathbf{v} = \langle u_1,u_2,u_3 \rangle + \langle v_1,v_2,v_3 \rangle = \langle u_1+v_1, u_2+v_2, u_3+v_3 \rangle$

$-\mathbf{u} = \langle -u_1, -u_2, -u_3 \rangle$ $\qquad c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle$

$\mathbf{0} = 0\mathbf{u} = \langle 0, 0, 0 \rangle$

Theorem A

For all vectors **u, v, w** and the real numbers a and b

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity of addition)
$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity " ")
$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u}$
$\mathbf{u} + -\mathbf{u} = \mathbf{0}$ additive identity $= \vec{0}$; additive inverse of $\vec{u}$ is $-\vec{u}$
$a(b\mathbf{u}) = (ab)\mathbf{u}$
$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
$(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ associativity + distributivity of scalar mult.
$1\mathbf{u} = \mathbf{u}$ 1 is the scalar mult. identity

$||\mathbf{u}|| = \sqrt{u_1^2+u_2^2+u_3^2}$

$||c\mathbf{u}|| = |c|\ ||\mathbf{u}||$ $|c|$ = abs. value of c

"magnitude/length of $\vec{u}$" (notation: $||\vec{u}||$, $|\vec{u}|$)

EX 5

Let $\mathbf{u} = \langle -1, 5, 2 \rangle$, find $||\mathbf{u}||$ and $||-3\mathbf{u}||$.
Also, find a vector, $\hat{\mathbf{u}}$ with the same direction as $\mathbf{u}$ but with magnitude = 1. (This is called a unit vector)

$$||\vec{u}|| = \sqrt{(-1)^2 + 5^2 + 2^2}$$

$$= \sqrt{30}$$

$$||-3\vec{u}|| = 3\sqrt{30}$$

notation:
$\hat{u}$ is a unit vector; a vector w/ length 1

$$\boxed{\hat{u} = \frac{\vec{u}}{||\vec{u}||}}$$

$$\hat{u} = \frac{\langle -1, 5, 2 \rangle}{\sqrt{30}} = \left\langle \frac{-1}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{2}{\sqrt{30}} \right\rangle$$