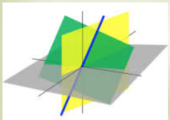
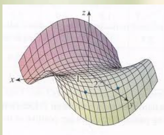


$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$



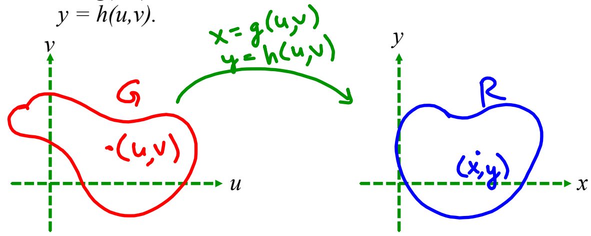
$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2}y\right]_{x=0}^{x=2y} dy = \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy = \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

Change of Variables (Jacobian Method)

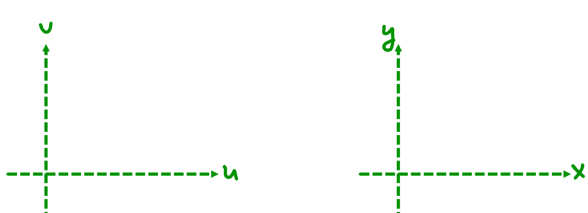
$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Transformations from a region G in the uv-plane to the region R in the xy-plane are done by equations of the form

$x = g(u,v)$
$y = h(u,v)$.



Ex 1 $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$, G is the rectangle given by $0 \le u \le 1$, $0 \le v \le 1$.



How is the integral of $f(x,y)$ over R related to the integral of $f(g(u,v),\, h(u,v))$ over G?

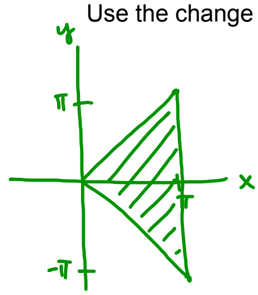
$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v),h(u,v))\left|J(u,v)\right|du\ dv$$

where $$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

EX 2 For polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, what is $J(r,\theta)$?

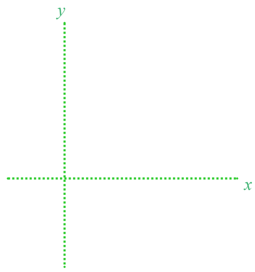
$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

EX 3 Evaluate $\iint_R \cos(x-y)\sin(x+y)\,dA$ where R is the triangle in the xy-plane with vertices at $(0,0)$, $(\pi, -\pi)$ and (π,π).
Use the change of variables to $u = x - y$ and $v = x + y$.



EX 4 Evaluate $\iint_R 5(x^2+y^2)\,dx\,dy$ where R is the region in Q_1 bounded by $x^2+y^2=9$, $x^2+y^2=16$, $y^2-x^2=1$, $y^2-x^2=9$.

Hint: Use $u=x^2+y^2$ and $v=y^2-x^2$ to transform R into a much nicer region (G).



Change of variables in 3 dimensions.

If $x=g(u,v,w)$
$y=h(u,v,w)$
$z=j(u,v,w)$

then

$$\iiint_R f(x,y,z)\,dx\,dy\,dz=\iiint_G f(g(u,v,w),h(u,v,w)\,j(u,v,w))\,|J(u,v,w)|\,du\,dv\,dw$$

where $J(u,v,w)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$

EX 5 Let's check the Jacobian for spherical coordinates.

$$x=\rho\cos\theta\sin\varphi$$
$$y=\rho\sin\theta\sin\varphi$$
$$z=\rho\cos\varphi$$

$$J(\rho,\theta,\varphi)=\begin{vmatrix}\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi}\\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi}\end{vmatrix}$$