If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

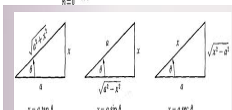
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

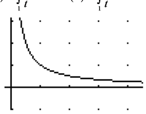
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x\frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

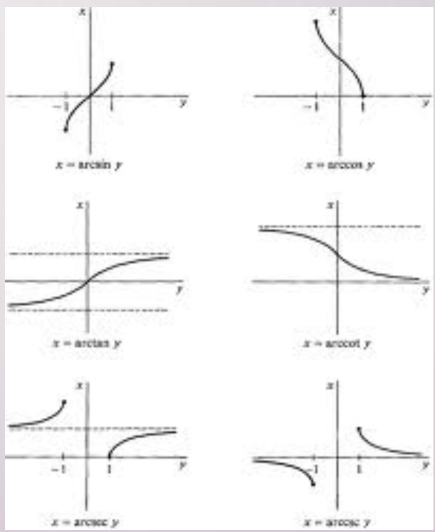
$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Inverse Trigonometry Functions and Their Derivatives



$$\frac{d}{dx}\arcsin x=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arccos x=\frac{-1}{\sqrt{1-x^2}}$$

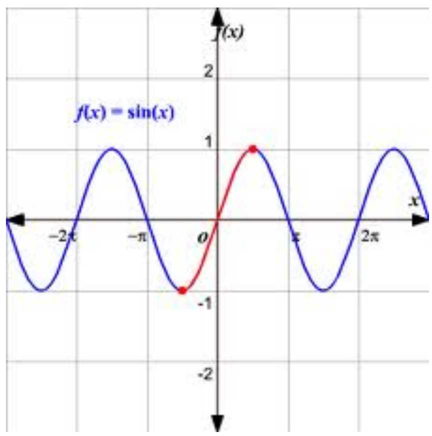
$$\frac{d}{dx}\arctan x=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\operatorname{arccot} x=\frac{-1}{1+x^2}$$

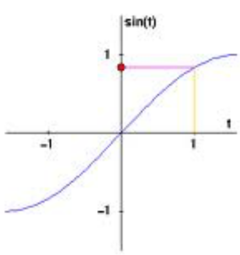
$$\frac{d}{dx}\operatorname{arcsec} x=\frac{1}{x\sqrt{x^2-1}}$$

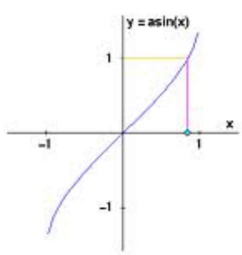
$$\frac{d}{dx}\operatorname{arccsc} x=\frac{-1}{x\sqrt{x^2-1}}$$

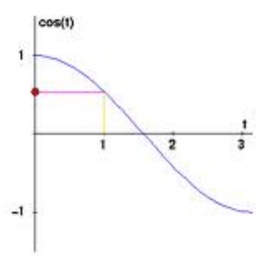
The graph of y = sin x does not pass the horizontal line test, so it has no inverse.

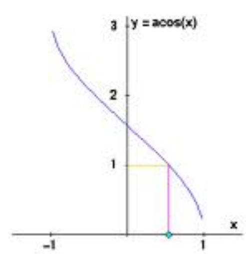


If we restrict the domain (to half a period), then we can talk about an inverse function.









notation: $\arcsin x = \sin^{-1} x = \operatorname{asin} x$

Definition

$x = \sin^{-1} y \iff y = \sin x \qquad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$x = \cos^{-1} y \iff y = \cos x \qquad x \in [0, \pi]$

(the angle returned by arcsin fn is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$)

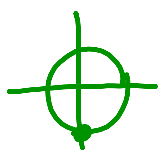
notation $\qquad \arccos(x) = \cos^{-1} x$

EX 1 Evaluate these without a calculator.

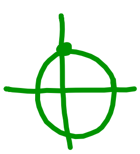
a) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

c) $\sin^{-1}(\sin(3\pi/2)) = \sin^{-1}(-1)$

$= -\frac{\pi}{2}$

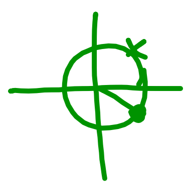


b) $\sin^{-1}(1) = \frac{\pi}{2}$

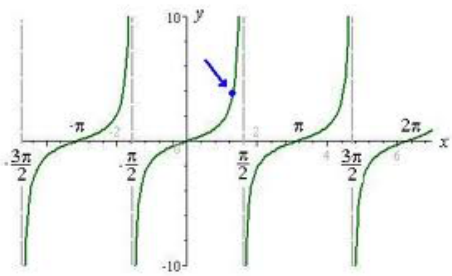


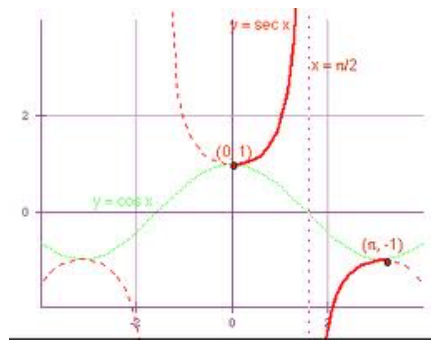
d) $\cos^{-1}(\cos(-\pi/4)) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\pi}{4}$



$y = tan\ x$ $\qquad\qquad$ $y = sec\ x$





<u>Definition</u>

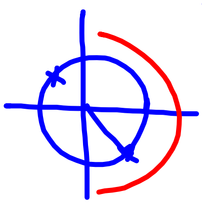
$$x = \tan^{-1} x \Leftrightarrow y = \tan x \qquad x \in (-\pi/2, \pi/2)$$

$$x = \sec^{-1} y \Leftrightarrow y = \sec x \qquad x \in [0, \pi/2) \cup (\pi/2, \pi]$$

EX 2 Evaluate without a calculator.

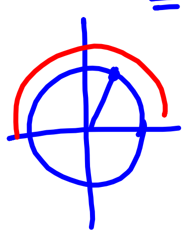
a) $\tan^{-1}(-1)$

$= -\frac{\pi}{4}$



b) $\sec^{-1}(2)$

$= \cos^{-1}\left(\frac{1}{2}\right)$



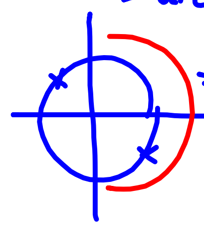
$= \frac{\pi}{3}$

$\sec\theta = 2$

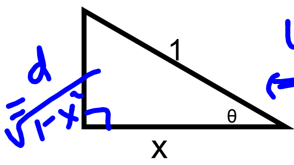
$\Leftrightarrow \cos\theta = \frac{1}{2}$

c) $\arctan(\tan(\frac{3\pi}{4}))$

$= \arctan(-1)$



$= -\frac{\pi}{4}$



let $\theta = \cos^{-1}x$

$\Leftrightarrow \cos\theta = x = \frac{x}{1}$

$\sin(\cos^{-1}x) =$

Pythagorean Thm.

$x^2 + d^2 = 1$

$d^2 = 1 - x^2$

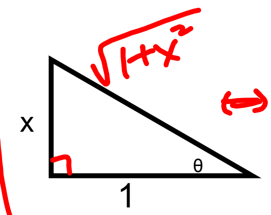
$d = \sqrt{1-x^2}$

$\sin\theta$

$= \frac{\sqrt{1-x^2}}{1}$

$= \sqrt{1-x^2}$

(θ acute)



$\theta = \tan^{-1}x$

$\Leftrightarrow \tan\theta = x = \frac{x}{1}$

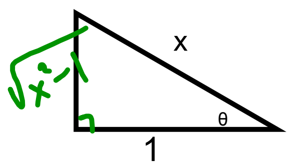
$\sec(\tan^{-1}x) =$

$\sec\theta$

$= \frac{\sqrt{1+x^2}}{1}$

$= \sqrt{1+x^2}$

(θ acute)



$\theta = \sec^{-1}x$

$\Leftrightarrow \sec\theta = x$

$\cos\theta = \frac{1}{x}$

$\tan(\sec^{-1}x) =$

$\tan\theta = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$

(θ acute)

EX 3 Calculate $sin[2cos^{-1}(1/4)]$ with no calculator.

$\sin(2\underbrace{\cos^{-1}(\tfrac{1}{4})}_{\theta})$

$= \sin(2\theta)$

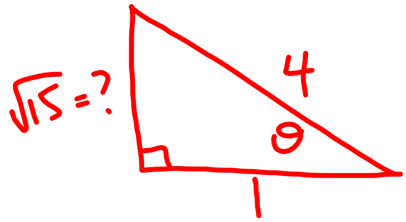
$= 2\sin\theta\cos\theta$

(by double angle identity)

$= 2\left(\frac{\sqrt{15}}{4}\right)\left(\frac{1}{4}\right) = \boxed{\frac{\sqrt{15}}{8}}$

let $\theta = \cos^{-1}(\tfrac{1}{4})$

$\cos\theta = \frac{1}{4}$



$?^2 + 1^2 = 4^2$

$?^2 = 15$

$D_x[\sin x] = \cos x \qquad D_x[\cos x] = -sinx$

$D_x[\tan x] = \sec^2 x \qquad D_x[\cot x] = -csc^2 x$

$D_x[\sec x] = \sec x tan x \qquad D_x[\csc x] = -\csc x cot x$

Derivatives of Inverse Trig Functions

Let $y = cos^{-1}x$. Find y'.

$y \in [0, \pi]$

$\Leftrightarrow \cos y = x$

$D_x(\cos y) = D_x(x)$

$-\sin y\,(y') = 1$

$y' = \frac{-1}{\sin y}$

$y' = \frac{-1}{\sqrt{1-x^2}} \Leftrightarrow \boxed{D_x(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}}$

$y = \cos^{-1}x$

$\Rightarrow \sin y = \sin(\cos^{-1}x)$

$= \sqrt{1-x^2}$

EX 4 $D_x[\tan^{-1}(5x^2-3x+1)]=$

$$= \frac{1(10x-3)}{1+(5x^2-3x+1)^2}$$

$D_x[\sin^{-1}x]=\frac{1}{\sqrt{1-x^2}},\ x\in(-1,1)$ $\quad$ $D_x[\cos^{-1}x]=\frac{-1}{\sqrt{1-x^2}}$

$D_x[\sec^{-1}x]=\frac{1}{|x|\sqrt{x^2-1}},\ |x|>1$ $\quad$ $D_x[\tan^{-1}x]=\frac{1}{1+x^2}$

EX 5 Evaluate these integrals.

a) $\int_{-1}^{1}\frac{1}{1+x^2}dx = \tan^{-1}x\Big|_{-1}^{1}$

$= \tan^{-1}1-\tan^{-1}(-1)$

$= \pi/4-(-\pi/4)=\boxed{\frac{\pi}{2}}$

b) $\int\frac{e^x}{1+e^{2x}}dx$

$$\int\frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}x+C$$

$$\int\frac{1}{1+x^2}dx=\tan^{-1}x+C$$

$$\int\frac{1}{x\sqrt{x^2-1}}dx=\sec^{-1}|x|+C$$

$u=e^x$

$du=e^x\,dx$

note: $e^{2x}=(e^x)^2$

$=\int\frac{du}{1+u^2}=\int\frac{1}{1+u^2}du$

$=\arctan u+C$

$=\boxed{\arctan(e^x)+C}$

<u>conclusion</u>

memorize (or write on a reference sheet)

formulas for inverse trig fn derivatives (+ integrals)