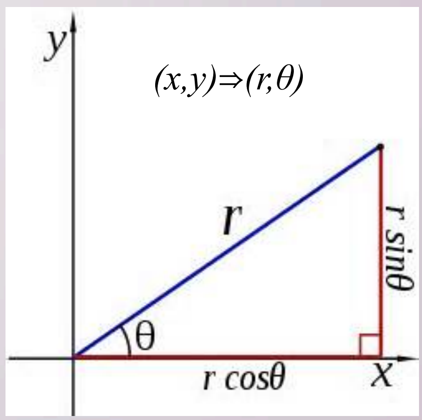
The Polar Coordinate System



If

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$$

or

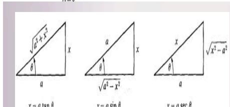
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

Then

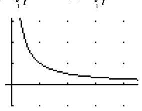
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f''''(x_0)}{4!}(x-x_0)^4 + \cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x) = \int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2) = \int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv - \int v\,du$$

the product rule for differentiation

where it comes from: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

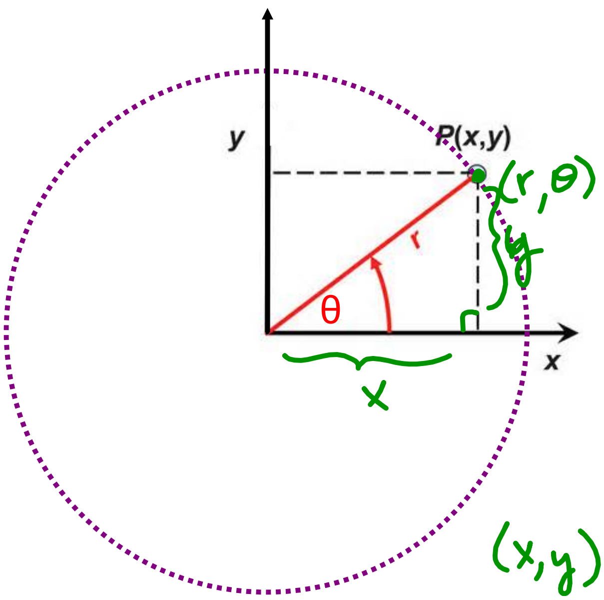
put into reverse $\int \frac{d}{dx}(uv) = \int (u\frac{dv}{dx} + v\frac{du}{dx})$

$uv = \int u\frac{dv}{dx} + \int v\frac{du}{dx}$

and then rearranged $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

The Polar Coordinate System

is a different way to express points in a plane.



by Pythagorean Thm

$x^2+y^2=r^2$

and $\tan\theta = \frac{y}{x}$

$(x,y) = (r\cos\theta, r\sin\theta)$

From Rectangular to Polar Coords	From Polar to Rectangular Coords
given (x,y)	given (r,θ)
then $\begin{cases} r=\sqrt{x^2+y^2} \\ \tan\theta = \frac{y}{x} \end{cases}$	then $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

EX 1 Find the rectangular coordinates for this point. *(4, π/6)*

$r \quad \theta$

find (x,y)

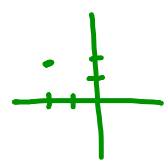
$x = r\cos\theta = 4\cos\left(\frac{\pi}{6}\right) = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$

$y = r\sin\theta = 4\sin\left(\frac{\pi}{6}\right) = 4\left(\frac{1}{2}\right) = 2$

$\boxed{(2\sqrt{3}, 2)}$

EX 2 Find the polar coordinates for this point. *(-2, 2)*

$x \quad y$



$r^2 = x^2 + y^2$

$r^2 = 4 + 4$

$r = \pm 2\sqrt{2}$

$r = 2\sqrt{2}$

$\tan\theta = \frac{2}{-2} = -1$

$\tan\theta = -1$

$\theta = 3\pi/4$

Note:

$\arctan(-1) = -\pi/4$

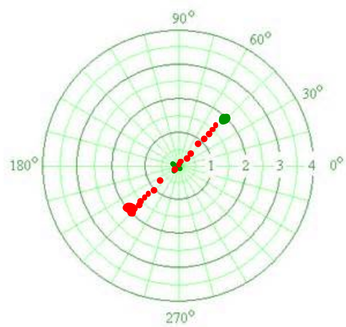
pt $\boxed{(2\sqrt{2}, 3\pi/4)}$

There are an infinite number of ways to write the same point in polar coordinates.

The point $(2, \pi/4)$ has other names.

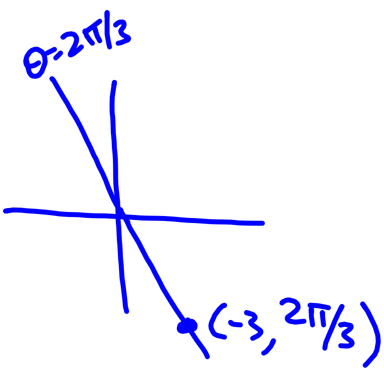
$\left(2, \frac{9\pi}{4}\right)$

$(-2, 5\pi/4)$



$(2, 5\pi/4)$

EX 3 Find three other ways to represent the polar coordinates for this point. $(-3, 2\pi/3)$



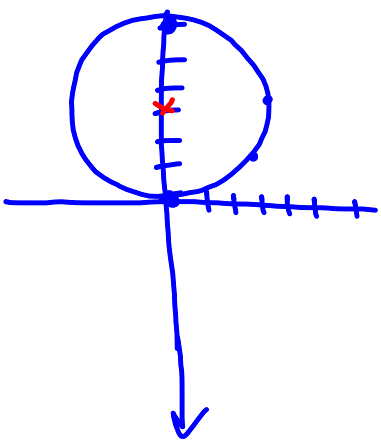
$(3, -\pi/3)$

$(3, 5\pi/3)$

$(-3, -4\pi/3)$

EX 4 Plot $r = 6\sin\theta$.

Prove that it is a circle in the Cartesian Coordinate system.



r	θ
0	0
6	$\pi/2$
0	π
-6	$3\pi/2$
$3\sqrt{2}$	$\pi/4$
3	$\pi/6$

same pts

same pts

(note: if $r=0$, $\theta \in \mathbb{R}$, then that's the origin)

$r = 6\sin\theta$

$r^2 = 6r\sin\theta$ (remember $r\sin\theta = y$)

$x^2+y^2 = 6y$ and $x^2+y^2=r^2$

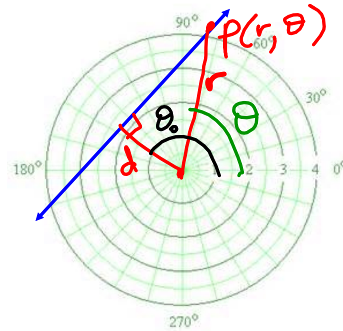
$x^2+(y^2-6y+9) = 0+9$

$x^2+(y-3)^2 = 9$

$x^2+(y-3)^2 = 3^2$ this is a circle centered at $(0,3)$ w/ radius 3.

Polar Equations

1) Lines

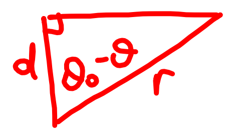


$P(r,\theta)$ a point on the line

$d = \perp$ distance to line (from origin)

θ_0 = angle to the $\perp$ line connecting the origin to line we want

extract the triangle:



$\cos(\theta_0-\theta) = \frac{d}{r}$

$\Leftrightarrow$ $\boxed{r = \dfrac{d}{\cos(\theta_0-\theta)}}$ eqn of line

compare w/:

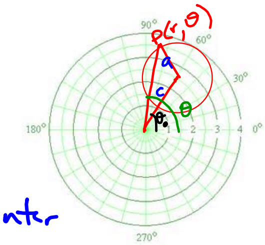
$y = mx+b$

m, b fixed constants

r, θ variables

d, θ_0 are fixed constants

2) Circles

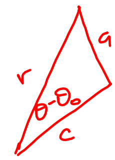


a = radius of circle

c = distance from origin to the center of circle

P = generic pt on circle $P(r, \theta)$

θ_0 = angle to the line connecting origin and center of circle.



(not a right triangle)

use law of cosines

$$\boxed{a^2 = r^2 + c^2 - 2rc\cos(\theta - \theta_0)}$$ circle eqn

r, θ variables

a, c, θ_0 are fixed

compare w/

$(x-h)^2 + (y-k)^2 = r^2$

x, y variables

h, k, r fixed constants

Special Cases:

If $c = a$ (circle goes through origin),

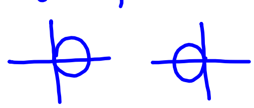
$$a^2 = r^2 + a^2 - 2ar\cos(\theta - \theta_0)$$
$$-a^2 \qquad\quad -a^2$$

$$0 = r^2 - 2ar\cos(\theta - \theta_0)$$

$$0 = r\left(r - 2a\cos(\theta - \theta_0)\right)$$

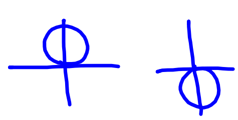
~~$r = 0$ one pt (graph)~~ or $\boxed{r = 2a\cos(\theta - \theta_0)}$ formula for circle that goes through the origin.

if $\theta_0 = 0$, $r = 2a\cos\theta$



if $\theta_0 = \pi/2$, we have $r = 2a\cos(\theta - \pi/2)$

$r = 2a\sin\theta$



3) Conics (Parabolas, Hyperbolas, Ellipses)

P on parabola

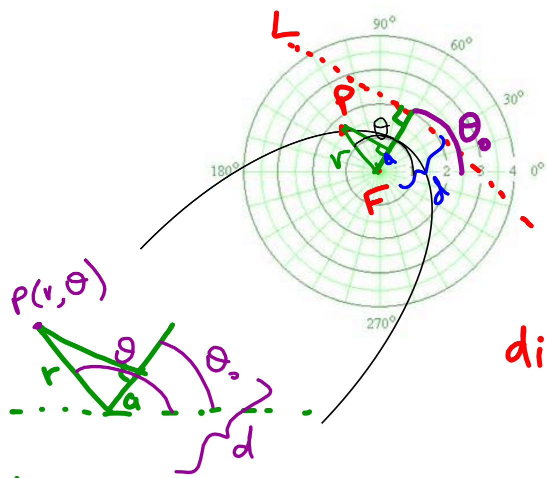
F focus (at the origin)

L is the directrix line

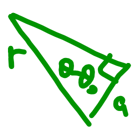
$$|\overline{PF}| = e\,|\overline{PL}|$$

e = eccentricity

$$e = \begin{cases} 1 & \text{parabola} \\ 0 < e < 1 & \text{ellipse} \\ e > 1 & \text{hyperbola} \end{cases}$$



$$|\overline{PF}| = r \quad \Rightarrow r = e|\overline{PL}| = e(d-a)$$



$$\cos(\theta-\theta_0) = \frac{a}{r} \Rightarrow a = r\cos(\theta-\theta_0)$$

$$r = e(d - r\cos(\theta-\theta_0))$$

$$r = ed - er\cos(\theta-\theta_0)$$

$$r(1 + e\cos(\theta-\theta_0)) = ed$$

$$\boxed{r = \frac{ed}{1+e\cos(\theta-\theta_0)}}$$

formula for conics in polar coords

r, θ variables

e, d, θ_0 fixed

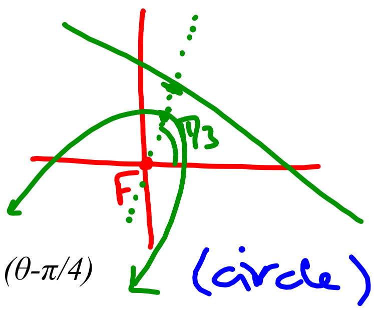
EX 5 Name the curve. If it is a conic, give its eccentricity and sketch it.

a) $r = \dfrac{2}{2+2\cos(\theta-\pi/3)}$ $\quad r = \dfrac{1}{1+\cos(\theta-\pi/3)}$ conic $\theta_0 = \pi/3$

$r = \dfrac{ed}{1+e\cos(\theta-\theta_0)}$

$e=1 \Rightarrow$ parabola

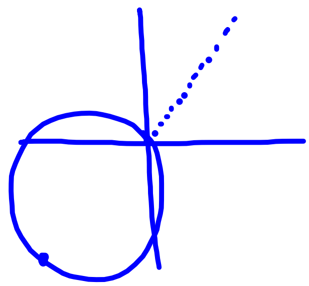
$ed=1 \Rightarrow d=1$



b) $r = -4\cos(\theta-\pi/4)$ (circle)

$\theta_0 = \pi/4$

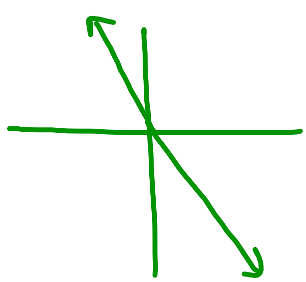
$r = 2a\cos(\theta-\theta_0)$
(circle thru origin)



r	θ
-4	$\pi/4$

c) $\theta = 2\pi/3$

radial line



Conclusion

intro to polar coords

- convert between rectangular / polar
- recognize simple graphs