If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

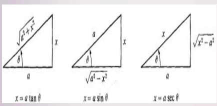
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

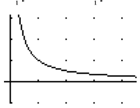
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$




$$\ln(x)=\int_1^x\frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt\cong 0.69315$$



$$\int u\,dv=uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

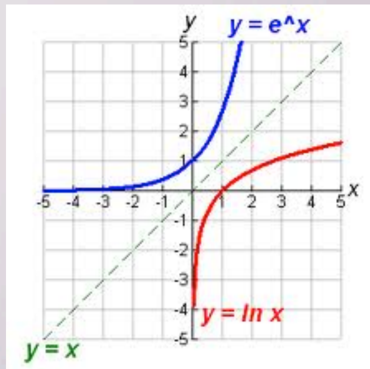
$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

# The Natural Logarithmic Function



The Natural Logarithmic Function

$$D_x\left(\frac{x^3}{3}\right) = x^2$$

RHS $\div x$

$$D_x\left(\frac{x^2}{2}\right) = x$$

$$D_x(x) = 1$$

$$\boxed{D_x(?) = \frac{1}{x}}$$

$$D_x\left(\frac{-1}{x}\right) = \frac{1}{x^2}$$

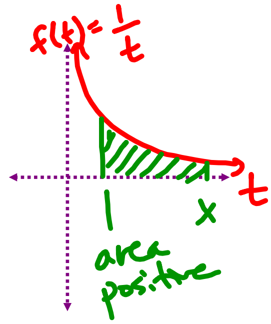
$$D_x\left(\frac{-1}{2x^2}\right) = \frac{1}{x^3}$$

There should be some function such that the derivative of that fn is $\frac{1}{x}$.

**Definition**

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0$$

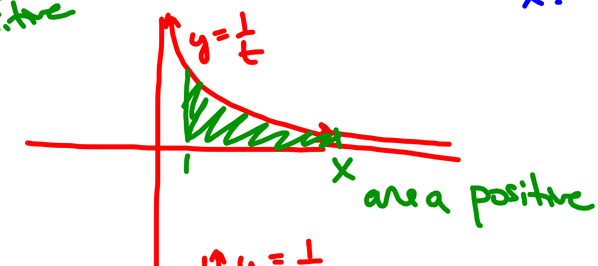
accumulation fn $\Rightarrow$

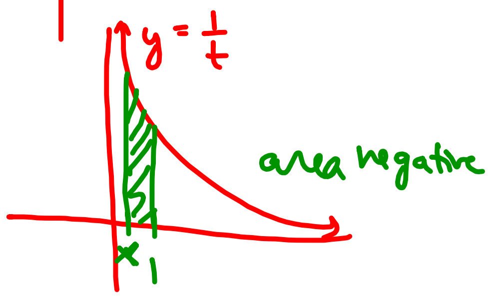



by defn
$\ln x$ = area under the curve $y = \frac{1}{t}$ from $t=1$ to $t=x$.

$\ln x > 0$ if $x > 1$

$\ln x < 0$ if $0 < x < 1$




From the First Fundamental Theorem of Calculus

$$D_x\left(\int_1^x \frac{1}{t}\,dt\right) = D_x(\ln x) = \frac{1}{x}, \quad x > 0$$

$$\boxed{D_x(\ln x) = \frac{1}{x}}, \quad x > 0$$

note: $D_x(\ln x) = \frac{1}{x}$ $(x>0)$

EX 1 Find $\frac{dy}{dx}$ if $y = ln\,(x^2)$

(use chain rule)

$$y' = \frac{dy}{dx} = \frac{1}{x^2}(2x) = \frac{2}{x}, \quad x>0$$

EX 2 Find $\frac{dy}{dx}$ and state the domain

a) $y = ln\left(\sqrt[3]{2x}\right) = \ln(2x)^{1/3}$ $(x>0)$

$$y' = \frac{1}{(2x)^{1/3}}\left(\frac{1}{3}(2x)^{-2/3}\right)(2) = \frac{2}{3(2x)} = \frac{1}{3x}$$

b) $y = \ln(3x^2 + 14x - 5)$ domain: $3x^2+14x-5>0$

$$y' = \frac{1}{3x^2+14x-5}(6x+14) = \frac{6x+14}{3x^2+14x-5}$$

---

$3x^2+14x-5>0$

$\boxed{(3x-1)(x+5)}>0$

critical values:

$x = -5, 1/3$

$-(-)$ $\quad -(+)$ $\quad +(+)$

$+$ $\quad -5 \quad$ $-$ $\quad \frac{1}{3} \quad$ $+$

sign line

test regions

answer: $\boxed{x<-5 \text{ or } x>1/3}$

domain

$$D_x\left[\ln|x|\right] = \frac{1}{x} \qquad x \neq 0$$

(we already stated by defn + First Fund. Thm of Calculus, $D_x(\ln x) = \frac{1}{x}$ if $x > 0$.)

Proof

Two cases

① $x > 0$.

$$D_x(\ln|x|) = D_x(\ln x) = \frac{1}{x} \checkmark.$$

② $x < 0$

$$D_x(\ln|x|) = D_x(\ln(-x)) = \frac{1}{-x} \cdot (-1) = \frac{1}{x} \checkmark$$

EX 3 Evaluate the integrals.

note: $D_x(\ln|x|) = \frac{1}{x}$, $x \neq 0$

$\int \frac{1}{x}\,dx = \ln|x| + C$

a) $\int \frac{6}{3x-2}\,dx = \int \frac{2}{u}\,du$

$u = 3x-2$

$2\,du = 3\,dx \cdot 2$

$2\,du = 6\,dx$

$= 2\int \frac{1}{u}\,du = 2(\ln|u|) + C$

$= \boxed{2\ln|3x-2| + C}$

b) $\int_2^5 \frac{3x}{7-2x^2}\,dx$ Note: This integral is valid because $7-2x^2 \neq 0$ on $[2,5]$

$u = 7-2x^2$

$du = -4x\,dx$

$-\frac{1}{4}du = x\,dx$

$x=2$, $u = 7-2\cdot 2^2$
$= 7-8 = -1$

$x=5$, $u = 7-2\cdot 5^2$
$= 7-50$
$= -43$

$= \int_{-1}^{-43} \frac{3(-\frac{1}{4})}{u}\,du$

$= \frac{-3}{4}\int_{-1}^{-43} \frac{1}{u}\,du = \frac{-3}{4}\ln|u|\Big|_{-1}^{-43}$

$= \frac{-3}{4}(\ln|-43| - \ln|-1|)$

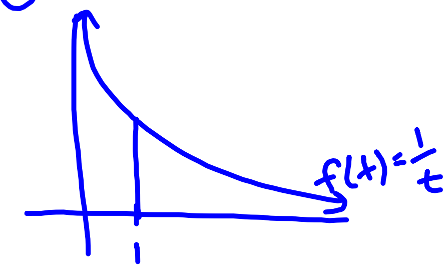
$= \frac{-3}{4}(\ln 43 - \ln 1)$

Properties of Logarithms

① $\ln 1 = 0$ ③ $\ln(ab) = \ln a + \ln b$

② $\ln a^r = r \ln a$ ④ $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

①




$\ln x = \int_1^x \frac{1}{t}\,dt$

$\ln 1 = \int_1^1 \frac{1}{t}\,dt$

$= 0$ (area)

Proof

$ax > 0$

③ $D_x(\ln(ax))$

$= \frac{1}{ax}(a) = \frac{1}{x}$

but also we know, by defn,

$D_x(\ln x) = \frac{1}{x}$

$\Rightarrow \ln x = \ln(ax) + c$ ($c$ some constant)

let $x=1$. Then we know $\ln 1 = 0$

$\ln 1 = \ln a + c$

$0 = \ln a + c$

$\Rightarrow c = -\ln a$

$\Rightarrow \ln x = \ln(ax) - \ln a$

$\Leftrightarrow \ln(ax) = \ln a + \ln x$ //

EX 4 Rewrite as a single logarithmic expression. ② $x>3$ and ③ $x>-3$

$\ln\left(x^2-9\right)-2\ln\left(x-3\right)-\ln\left(x+3\right)$ domain: $x>3$

$= \ln(x^2-9) + \ln(x-3)^{-2} + \ln(x+3)^{-1}$

$= \ln\left((x^2-9)(x-3)^{-2}(x+3)^{-1}\right) = \ln\left(\frac{\cancel{(x-3)}\cancel{(x+3)}}{(x-3)^{\cancel{2}}\cancel{(x+3)}}\right)$

$= \ln\left(\frac{1}{x-3}\right) = \boxed{-\ln(x-3)}$

EX 5 Find $\frac{dy}{dx}$ by logarithmic differentiation $y=\frac{\left(x^2+3\right)^{\frac{2}{3}}\left(3x+2\right)^2}{\sqrt{x+1}}$

① take ln of both sides.

② do implicit differentiation.

$\ln y = \ln\left(\frac{(x^2+3)^{2/3}(3x+2)^2}{\sqrt{x+1}}\right)$

$\ln y = \frac{2}{3}\ln(x^2+3) + 2\ln(3x+2) - \frac{1}{2}\ln(x+1)$

$D_x(\ln y) = D_x\left(\frac{2}{3}\ln(x^2+3) + 2\ln(3x+2) - \frac{1}{2}\ln(x+1)\right)$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{3}\left(\frac{2x}{x^2+3}\right) + 2\left(\frac{(3)}{3x+2}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)$

$\frac{dy}{dx} = y\left(\frac{4x}{3(x^2+3)} + \frac{6}{3x+2} - \frac{1}{2(x+1)}\right)$

$\frac{dy}{dx} = \frac{(x^2+3)^{2/3}(3x+2)^2}{\sqrt{x+1}}\left(\frac{4x}{3(x^2+3)} + \frac{6}{3x+2} - \frac{1}{2(x+1)}\right)$

"Take Home message"

$$D_x\left(\ln|x|\right) = \frac{1}{x}, \quad x \neq 0$$

$$\Longleftrightarrow \quad \int \frac{1}{x}\,dx = \ln|x| + C \qquad (x \neq 0)$$

Warning:

(1) $\displaystyle\int \frac{1}{\text{Joe}}\,d(\text{Joe}) = \ln|\text{Joe}| + C$

(2) $\displaystyle\int \frac{1}{y^2}\,d(y^2) = \ln|y^2| + C$

(3) $\displaystyle\int \frac{1}{\underline{\underline{y}}^2}\,d\underline{\underline{y}} \neq \ln|y^2| + C$

$$\int \frac{1}{y^2}\,dy = \int y^{-2}\,dy = \frac{y^{-1}}{-1} + C$$

(4) $\displaystyle\int \frac{1}{\underline{\underline{\heartsuit}}}\,d\underline{\underline{\heartsuit}} = \ln|\heartsuit| + C$