If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

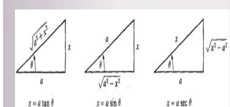
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

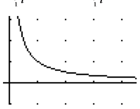
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x\frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Infinite Series

$$S_\infty=\sum_{n=1}^{\infty}9(0.1)^n = .9999999\ldots\ldots = .\overline{9} = 1$$

$$S_\infty=\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots = 1$$

Zeno's Paradox says that if you step from 0 to 1/2, then keep taking steps halfway between where you are and 1 that you will never get to 1.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}$$

Let S_i be the partial sum of the first i terms in the sequence.

$$S_1=\frac{1}{2}= \frac{1}{2} = 1-\frac{1}{2}$$

$$S_2=\frac{1}{2}+\frac{1}{4} = \frac{3}{4} = 1-\frac{1}{4} = 1-\frac{1}{2^2}$$

$$S_3=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}= \frac{7}{8} = 1-\frac{1}{8} = 1-\frac{1}{2^3}$$

$$S_4 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = \frac{15}{16} = 1-\frac{1}{16} = 1-\frac{1}{2^4}$$

$$\vdots$$

$$S_n=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}= 1-\frac{1}{2^n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\underbrace{\frac{1}{2^n}}_{0}\right)= 1-0 = 1$$

Infinite Series $\qquad a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \boxed{\sum_{i=1}^{\infty} a_i}$

a series is the sum of a sequence

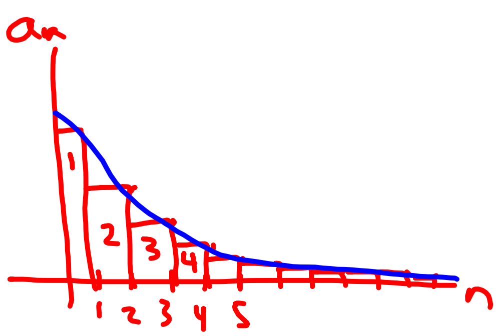
Partial Sum $\qquad \sum_{i=1}^{n} a_i = S_n$

Definition

$\sum a_i$ converges and has a sum, S, if the sequence of partial sums converges to S, i.e. $\lim_{n \to \infty} S_n = S$.

If $\{S_n\}$ diverges, then the series diverges and has no sum.

the sequence of partial sums



$a_1 + a_2 + a_3 + \ldots.$

area of rectangle 1

$= a_1 (1) = a_1$

Geometric Series

$a \neq 0$ $$\sum_{i=1}^{\infty} ar^{i-1} = a + ar + ar^2 + ar^3 + \dots$$

EX 1 Show that a geometric series converges for at least some r and find its sum.

nth partial sum

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \dots + \cancel{ar^n}$$

$$- \quad rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \dots \cancel{+} ar^{n+1}$$

$$S_n - rS_n = a - ar^{n+1}$$

$$S_n(1-r) = a(1-r^{n+1})$$

$$\boxed{S_n = \frac{a(1-r^{n+1})}{1-r}}$$ sum of the first n terms of a <u>geometric series</u>

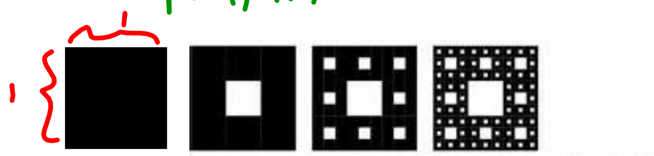
we know $r^{n+1} \to 0$ as $n \to \infty$ whenever $|r|<1$.

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r} = \boxed{\frac{a}{1-r} \text{ for all } r \text{ such that } |r|<1}$$

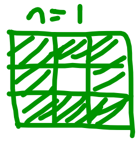
$\therefore$ A Geometric series converges to $S = \frac{a}{1-r}$ if $|r|<1$, otherwise it diverges.

EX 2 If this pattern continues indefinitely, what fraction of the original square will eventually be unshaded?

recursive pattern



n=0 n=1 n=2 n=3



want to add up all unshaded areas

$a_1 = \frac{1}{9}$ $\quad a_2 = \frac{1}{9} + 8\left(\frac{1}{9}\right)^2$ $\quad a_3 = \frac{1}{9} + 8\left(\frac{1}{9}\right)^2 + 64\left(\frac{1}{9}\right)^3$

$$a_3 = \frac{1}{9} + 8\left(\frac{1}{9}\right)^2 + 8^2\left(\frac{1}{9}\right)^3$$

$$a_4 = \frac{1}{9} + 8\left(\frac{1}{9}\right)^2 + 8^2\left(\frac{1}{9}\right)^3 + 8^3\left(\frac{1}{9}\right)^4$$

$$= \frac{1}{9} + \left(\frac{8}{9}\right)\left(\frac{1}{9}\right) + \left(\frac{8^2}{9^2}\right)\left(\frac{1}{9}\right) + \left(\frac{8^3}{9^3}\right)\left(\frac{1}{9}\right)$$

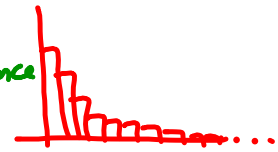
$$= \underset{n=1}{\frac{1}{9}} + \underset{n=2}{\frac{1}{9}\left(\frac{8}{9}\right)} + \underset{n=3}{\frac{1}{9}\left(\frac{8}{9}\right)^2} + \frac{1}{9}\left(\frac{8}{9}\right)^3$$

$$\underbrace{a = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{8}{9}\right)^{n-1}}_{\text{geom. series w/ } r = \frac{8}{9} < 1}$$

$$a = \frac{1/9}{1 - 8/9} = \frac{1/9}{1/9} = 1$$

Theorem

nth term test for $\boxed{\text{divergence}}$ ⁂ never tests for convergence



① If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

If A, then B

② ⟺ $\boxed{\text{If } \lim_{n\to\infty} a_n \neq 0 \text{ or if } \lim_{n\to\infty} a_n \text{ DNE, then the series diverges.}}$ ⟺ if not B, then not A.

Pf Let S_n = partial sum, and $S = \lim_{n\to\infty} S_n$
(we know S is finite because we were told the series converges.) $\Rightarrow S_n - S_{n-1} = (a_1 + \dots + a_n) - (a_1 + \dots + a_{n-1}) = a_n$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = S - S = 0 \ \#$$

EX 3 Does $\sum_{i=1}^{\infty} \frac{3i-7}{4i+3}$ converge or diverge?

try nth term test (for divergence):

$$\lim_{n\to\infty} \frac{3n-7}{4n+3} = \frac{3}{4} \neq 0 \Rightarrow \sum_{i=1}^{\infty} \frac{3i-7}{4i+3} \text{ diverges}$$

Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n} =$

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

but does it converge?

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

$$= 1 + \frac{1}{2} + \overbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}^{> 1/2} + \overbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}^{> 1/2}$$

$$+ \overbrace{\left(\frac{1}{9} + \ldots + \frac{1}{16}\right)}^{> 1/2} + \ldots + \frac{1}{n}$$

$$S_n > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{n}$$

$$\Rightarrow \lim_{n \to \infty} S_n > \lim_{n \to \infty} \Big(\underbrace{1 + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{n}}_{\text{keeps growing w/ } n} \Big)$$

$\Rightarrow$ harmonic series diverges.

This is a great example to convince you that $\lim_{n \to \infty} a_n = 0$ does <u>not</u> imply convergence of $\sum a_i$.

EX 4 does $\sum_{i=1}^{\infty} \frac{3}{i(i+1)}$ converge or diverge?

(hint: need PFD)

$$\sum_{i=1}^{\infty} \frac{3}{i(i+1)} = \sum_{m=1}^{\infty} \frac{3}{m(m+1)}$$

$$\frac{3}{m(m+1)} = \frac{A}{m} + \frac{B}{m+1}$$

$$3 = A(m+1) + Bm$$

$m=0$: $3 = A$

$m=-1$: $3 = -B$

$B = -3$

$$\sum_{m=1}^{\infty} \frac{3}{m(m+1)}$$

$$= \sum_{m=1}^{\infty} \left(\frac{3}{m} - \frac{3}{m+1}\right)$$

(collapsing or telescoping sum)

$$S_n = \left(\frac{3}{1} - \cancel{\frac{3}{2}}\right) + \left(\cancel{\frac{3}{2}} - \cancel{\frac{3}{3}}\right) + \left(\cancel{\frac{3}{3}} - \cancel{\frac{3}{4}}\right) + \left(\cancel{\frac{3}{4}} - \cancel{\frac{3}{5}}\right) + \dots + \left(\cancel{\frac{3}{n}} - \frac{3}{n+1}\right)$$

$$S_n = 3 - \frac{3}{n+1}$$

to find the infinite sum, let $n \to \infty$.

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(3 - \underbrace{\frac{3}{n+1}}_{0}\right)$$

$$= 3$$

$\Rightarrow$ series converges

Linearity of a Convergent Positive Series

If $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ both converge,

then $\sum_{i=1}^{\infty} ca_i = c\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty}(a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i$

also converge.

EX 5 Does $\sum_{k=1}^{\infty}\left[5(\tfrac{1}{2})^k - 3(\tfrac{1}{7})^k\right]$ diverge or converge?

notice: $\sum_{k=1}^{\infty} 5(\frac{1}{2})^k$ is geom. series w/ $r = \frac{1}{2} < 1$

$\Rightarrow$ it converges

likewise $\sum_{k=1}^{\infty} 3(\frac{1}{7})^k$ also converges.

(it's geom. w/ $r = \frac{1}{7} < 1$)

$\Rightarrow \sum_{k=1}^{\infty}\left(5(\frac{1}{2})^k - 3(\frac{1}{7})^k\right)$ converges also

Theorem

If $\sum_{k=1}^{\infty} a_k$ diverges and c ≠ 0, then $\sum_{k=1}^{\infty} ca_k$ diverges.

Grouping Terms in an infinite series

The terms in a convergent positive series can be grouped in any way and the new series will still converge to the same sum.

but we cannot re-group terms of divergent series

Why don't we just use computers to tell if a series converges?

Consider the Harmonic Series:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$$

On a computer, for $n=10^{43}$, $S_n = 100$ and $S_{272,000,000} \cong 20$.

$\Rightarrow$ the sum grows very slowly!

In Conclusion

In general, we are now trying to decide if an infinite series converges (adds to a finite value) or diverges (adds to ∞).

Tests so far:

1. look for a geometric series.
2. n^{th} term test for divergence.

⋮

(*) argue by partial sums.