If

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

or

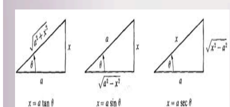
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

Then

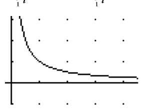
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &\quad + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \frac{f''''(x_0)}{4!}(x - x_0)^4 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \end{aligned}$$



$$\ln(x) = \int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2) = \int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv - \int v\,du$$

where it comes from:

the product rule for differentiation

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

put into reverse

$$\int \frac{d}{dx}(uv) = \int \left(u\frac{dv}{dx} + v\frac{du}{dx}\right)$$

$$uv = \int u\frac{dv}{dx} + \int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$$

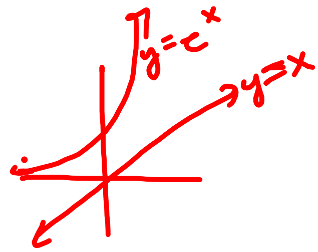
Other Indeterminate Forms

Indeterminate forms

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad 0\cdot\infty,\quad \infty-\infty,\quad 0^0,\quad \infty^0,\quad 1^\infty$$

Other Indeterminate Forms $\frac{\infty}{\infty}$

$$\lim_{x\to\infty}\frac{x}{e^x} = 0$$



$$\lim_{x\to\infty}\frac{x^2+4x-5}{3x^2-2x} = \lim_{x\to\infty}\frac{x^2}{3x^2} = \lim_{x\to\infty}\frac{1}{3} = \frac{1}{3}$$

L'Hopital's rule for $\frac{\infty}{\infty}$ $\left(\frac{\pm\infty}{\pm\infty}\right)$

If $\lim\limits_{x\to a}|f(x)| = \lim\limits_{x\to a}|g(x)| = \infty$ then

$\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ assuming $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ exists (finite or ∞).

Now use l'Hopital's Rule for this:

$$\lim_{x\to\infty}\frac{x}{e^x} \overset{L}{=} \lim_{x\to\infty}\frac{1}{e^x} = 0$$

$\left(\frac{\infty}{\infty}\text{ case}\right)$

EX 1 $\lim_{x\to\infty}\frac{x^9}{e^x}$ $\stackrel{L}{=} \lim_{x\to\infty}\frac{9x^8}{e^x} \stackrel{L}{=} \lim_{x\to\infty}\frac{72x^7}{e^x}$

$\left(\frac{\infty}{\infty} \text{ case}\right)$

$\stackrel{L}{=} \cdots \stackrel{L}{=} \lim_{x\to\infty}\frac{\text{const.}}{e^x} = 0$

EX 2 $\lim_{x\to\infty}\frac{(\ln x)^2}{2^x}$ $\stackrel{L}{=} \lim_{x\to\infty}\frac{2(\ln x)\left(\frac{1}{x}\right)}{2^x \ln 2}$

$\left(\frac{\infty}{\infty} \text{ case}\right)$

$= \lim_{x\to\infty}\frac{2}{\ln 2}\left(\frac{\ln x}{x2^x}\right)$

$\stackrel{L}{=} \lim_{x\to\infty}\frac{2}{\ln 2}\left(\frac{1/x}{2^x + x2^x\ln 2}\right)$

$= \lim_{x\to\infty}\frac{2}{\ln 2}\left(\frac{1}{x2^x + x^2 2^x \ln 2}\right)$

$= 0$

Indeterminate Forms $0 \cdot \infty$ and $\infty - \infty$

EX 3 $\lim_{x\to 0} 3x^2 \csc^2 x$ $= \lim_{x\to 0} \frac{3x^2}{\sin^2 x} \overset{L}{=} \lim_{x\to 0} \frac{6x}{2\sin x \cos x}$

($0\cdot\infty$ case) $\left(\frac{0}{0}\text{ case}\right)$ $\overset{L}{=} \lim_{x\to 0} \frac{6}{2\cos x\cos x + 2\sin x(-\sin x)}$

$= \frac{6}{2(1)+0} = 3$

Note: we cannot use L'Hopital's Rule yet, even though it's indeterminate case; so we must algebraically manipulate this to be $\frac{0}{0}$ or $\frac{\infty}{\infty}$ case.

EX 4 $\lim_{x\to \pi/2} (\tan x - \sec x)$

($\infty - \infty$ case) indeterminate

$= \lim_{x\to \pi/2} \left(\frac{\sin x}{\cos x} - \frac{1}{\cos x}\right)$

$= \lim_{x\to \pi/2} \left(\frac{\sin x - 1}{\cos x}\right)$

$\left(\frac{0}{0}\text{ case}\right)$ $\overset{L}{=} \lim_{x\to \pi/2} \frac{\cos x}{-\sin x} = \frac{0}{-1} = \boxed{0}$

Indeterminate Forms 0^0, ∞^0, 1^∞

Note:

$f(x) = e^{\ln f(x)}$

EX 5 $\lim_{x\to 0}(\cos x)^{1/x^2} = \lim_{x\to 0} e^{\ln(\cos x)^{1/x^2}}$

(1^∞ case) indeterminate

$= \lim_{x\to 0} e^{\frac{1}{x^2}\ln(\cos x)}$

$= \lim_{x\to 0} e^{\frac{\ln(\cos x)}{x^2}} = e^{\lim_{x\to 0}\frac{\ln(\cos x)}{x^2}}$ ($\frac{0}{0}$ case)

$\overset{L}{=} e^{\lim_{x\to 0}\frac{\frac{-\sin x}{\cos x}}{2x}} = e^{\lim_{x\to 0}\frac{-\tan x}{2x}}$

$\overset{L}{=} e^{\lim_{x\to 0}\frac{-\sec^2 x}{2}} = \boxed{e^{-1/2}}$

EX 6 $\lim_{x\to 0^+} x^x$

(0^0 case)

$= \lim_{x\to 0^+} e^{\ln x^x}$

$= \lim_{x\to 0^+} e^{x\ln x} = \lim_{x\to 0^+} e^{\frac{\ln x}{1/x}} = e^{\lim_{x\to 0^+}\frac{\ln x}{1/x}}$ ($\frac{\infty}{\infty}$ case)

$\overset{L}{=} e^{\lim_{x\to 0^+}\frac{1/x}{-1/x^2}}$

$= e^{\lim_{x\to 0^+}-\frac{x^2}{x}} = e^{\lim_{x\to 0^+}-x} = e^0 = \boxed{1}$

Indeterminate Forms

$\frac{0}{0}$

$\frac{\pm\infty}{\pm\infty}$

$0 \cdot \infty$

$\infty - \infty$

0^0

∞^0

1^∞

Note: we can only use L'Hopital's Rule for $\frac{0}{0}$ or $\frac{\infty}{\infty}$ case!!!!

NOT Indeterminate Forms

$1^0 = 1$

$0^\infty = 0$

$\left.\begin{array}{l} \infty^\infty \\ \infty + \infty \\ \infty \cdot \infty \end{array}\right\} \infty$

$\frac{0}{\infty} = 0$

$\frac{\infty}{0} = \infty$

ex $\lim_{x\to 0} (\sin x)^{1/x^2} = 0$

(0^∞ case)

ex $\lim_{x\to \frac{\pi}{2}^-} (\tan x + \sec x) = \infty$

($\infty + \infty$ case)