If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

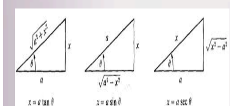
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

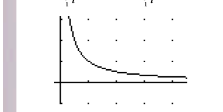
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots = \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$



$$\ln(x)=\int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Indeterminate Forms of Type 0/0

Indeterminate forms

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad 0\cdot\infty,\quad \infty-\infty,\quad 0^{\infty},\quad \infty^{0},\quad 1^{\infty}$$

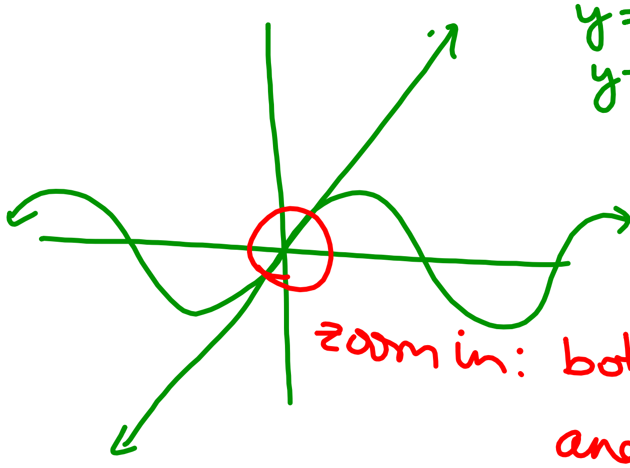
Previously we found the limit of an expression which appeared to approach $\frac{0}{0}$.

Determine this limit. $\lim_{x\to 2}\frac{x^2-2x}{x^2+3x-10} = \lim_{x\to 2}\frac{\cancel{(x-2)}x}{\cancel{(x-2)}(x+5)}$

$\left(\frac{0}{0}\text{ case}\right)$

$= \lim_{x\to 2}\frac{x}{x+5} = \frac{2}{7}$

We also were able to geometrically determine this limit. $\lim_{x\to 0}\frac{\sin x}{x}$ $\left(\text{also } \frac{0}{0}\text{ case}\right)$



$y=\sin x$
$y=x$

$\lim_{x\to 0}\frac{\sin x}{x} = 1$

zoom in: both $y=\sin x$ graph and $y=x$ graph approach 0 (at $x=0$) at same rate

(ie. their slopes are basically the same when $x \approx 0$)

L'Hopital's Rule:

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ and $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists

(either finite or ±∞), then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

(notice: this is useful only for $\frac{0}{0}$ case)

This makes both of the previous problems more simple.

EX 1 Determine these limits using the rule above.

a) $\lim_{x \to 2} \frac{x^2 - 2x}{x^2 + 3x - 10}$

$\overset{L}{=} \lim_{x \to 2} \frac{2x-2}{2x+3}$

$= \frac{2}{7}$

b) $\lim_{x \to 0} \frac{\sin x}{x} \overset{L}{=} \lim_{x \to 0} \frac{\cos x}{1}$

$= 1$

EX 2 $\lim_{x\to 0}\dfrac{e^x - e^{-x}}{2\sin x} \overset{L}{=} \lim_{x\to 0}\dfrac{e^x-(-e^{-x})}{2\cos x}$

$\left(\frac{0}{0}\text{ case}\right)$

$= \dfrac{1+1}{2(1)} = 1$

EX 3 $\lim_{x\to 0^+}\dfrac{7^{\sqrt{x}}-1}{2^{\sqrt{x}}-1} \overset{L}{=} \lim_{x\to 0^+}\dfrac{7^{\sqrt{x}}(\ln 7)\cancel{\left(\frac{1}{2\sqrt{x}}\right)}}{2^{\sqrt{x}}(\ln 2)\cancel{\left(\frac{1}{2\sqrt{x}}\right)}}$

$\left(\frac{0}{0}\text{ case}\right)$

$= \lim_{x\to 0^+}\left(\dfrac{\ln 7}{\ln 2}\right)\left(\dfrac{7}{2}\right)^{\sqrt{x}} = \dfrac{\ln 7}{\ln 2}$

EX 4 $\lim_{x\to 0}\frac{\sin x - \tan x}{x^2 \sin x} = \lim_{x\to 0}\frac{\cancel{\sin x}\left(1-\frac{1}{\cos x}\right)}{x^2\cancel{\sin x}}$

$\left(\frac{0}{0} \text{ case}\right)$

$$= \lim_{x\to 0}\frac{1-\sec x}{x^2} \overset{L}{=} \lim_{x\to 0}\frac{-\sec x\tan x}{2x}$$

$$\overset{L}{=} \lim_{x\to 0}\frac{(-\sec x\tan x)\tan x + (-\sec x)\sec^2 x}{2}$$

$$= \frac{0-1}{2} = -\frac{1}{2}$$

EX 5 $\lim_{x\to 0}\frac{\cos x}{x}$

$\left(\frac{1}{0} \text{ case}\right)$ NOTE: we cannot use L'Hopital's Rule here (this is not an indeterminate case)

We know the $\frac{1}{0}$ case turns into ∞ or $-\infty$ or DNE

We need to know what happens as $x\to 0^+$, and as $x\to 0^-$.

$\frac{\cos x}{x}$: $x\to 0^-$, $\frac{+}{-}$ $\lim_{x\to 0^-}\frac{\cos x}{x} = -\infty$

$\frac{\cos x}{x}$: $x\to 0^+$, $\frac{+}{+}$ $\lim_{x\to 0^+}\frac{\cos x}{x} = \infty$

$\Rightarrow \lim_{x\to 0}\frac{\cos x}{x}$ DNE

EX 6 $\lim_{x\to 0^+} \dfrac{\int_0^x \sqrt{t}\cos t\,dt}{x^2}$ $\overset{L}{=} \lim_{x\to 0^+} \dfrac{\sqrt{x}\cos x}{2x}$

$\left(\frac{0}{0}\text{ case}\right)$

$= \lim_{x\to 0^+} \dfrac{\cos x}{2\sqrt{x}}$ $\left(\frac{1}{0}\text{ case}\right)$

$= \infty$

EX 7 $\lim_{x\to 0^-} \dfrac{\sin x + \tan x}{e^x + e^{-x} - 2}$ $\overset{L}{=} \lim_{x\to 0^-} \dfrac{\cos x + \sec^2 x}{e^x - e^{-x}}$ $\left(\frac{2}{0}\text{ case}\right)$

$= \boxed{-\infty}$

$\left(\frac{0}{0}\text{ case}\right)$

$\frac{+}{-}$

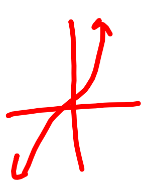
<u>ex</u> $e^x - e^{-x}$

try $x = -0.01$

$e^{-0.01} - e^{0.01}$

$e^{0.01} > e^{-0.01}$

we could also have remembered $\sinh x = \dfrac{e^x - e^{-x}}{2}$



$y = \sinh x$

In Conclusion

Use L'Hopital's Rule when we have the $\frac{0}{0}$ case for a limit. (indeterminate)