Trigonometric Integrals

a) $\sin x \cos y = \frac{1}{2}[\sin(x-y)+\sin(x+y)]$

b) $\sin x \sin y = \frac{1}{2}[\cos(x-y)-\cos(x+y)]$

c) $\cos x \cos y = \frac{1}{2}[\cos(x-y)+\cos(x+y)]$

If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

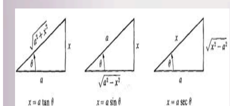
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

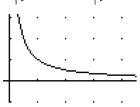
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv - \int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

$$\int \frac{d}{dx}(uv)=\int \left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Trigonometric Integrals

Combining u-substitution and the trigonometric identities, we will address three forms of these integrals.

1. $\int \sin^n x\,dx, \quad \int \cos^n x\,dx$
2. $\int \sin^m x \cos^n x\,dx$
3. $\int \sin(mx)\cos(nx)\,dx, \quad \int \sin(mx)\sin(nx)\,dx, \quad \int \cos(mx)\cos(nx)\,dx$

EX 1 $\int \sin^3 x\,dx$

$= \int \sin^2 x(\sin x\,dx)$

$= \int (1-\cos^2 x)\sin x\,dx$

$= \int \sin x\,dx - \int \cos^2 x \sin x\,dx$

$= -\cos x - \int \cos^2 x \sin x\,dx$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

Type 1

If n is odd,

use $\sin^2 x + \cos^2 x = 1$.

If n is even,

use half-angle formulas.

$\sin^2 x = \dfrac{1-\cos 2x}{2}$

$\cos^2 x = \dfrac{1+\cos 2x}{2}$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$= -\cos x - -\int u^2\,du$

$= -\cos x + \left(\frac{u^3}{3}\right) + C$

$= \boxed{-\cos x + \frac{1}{3}\cos^3 x + C}$

EX 2 $\int \cos^4 x\,dx = \int (\cos x)^4\,dx$

$= \int \cos^2 x \cos^2 x\,dx$

$= \int \left(\frac{1+\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx$

$= \frac{1}{4}\int \left(1+2\cos(2x)+\cos^2(2x)\right)dx$

$= \frac{1}{4}\left(x + 2\int \cos(2x)\,dx\right) + \frac{1}{4}\int \cos^2(2x)\,dx$

Type 1 again (n even)

Type 1
If n is odd,
use $\sin^2 x + \cos^2 x = 1$.
If n is even,
use half-angle formulas.
$\sin^2 x = \frac{1-\cos 2x}{2}$
$\cos^2 x = \frac{1+\cos 2x}{2}$

$\int \cos(2x)\,dx$

$u = 2x$

$du = 2\,dx$

$\frac{1}{2}du = dx$

$= \int \frac{\cos u}{2}\,du$

$= \frac{1}{2}\sin u + C$

$= \frac{1}{2}\sin(2x) + C$

Conclusion: (Shortcut)

$\int \sin(mx+b)\,dx = -\frac{\cos(mx+b)}{m} + C$

and $\int \cos(mx+b)\,dx = \frac{\sin(mx+b)}{m} + C$

$= \frac{1}{4}x + \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right) + \frac{1}{4}\int \frac{1+\cos(4x)}{2}\,dx$

$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}\int (1+\cos(4x))\,dx$

$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}\left(x + \frac{\sin(4x)}{4}\right) + C$

$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C$

$= \boxed{\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C}$

EX 3 $\int \underbrace{\cos^5}\, x \sin^{-4} x\, dx$

Type 2
If m or n is odd and positive, factor out $\sin x$ *or* $\cos x$ *and use* $\sin^2 x + \cos^2 x = 1$.
If m and n are even and positive, use half – angle identities.

$$= \int \cos x (\cos^4 x)(\sin x)^{-4}\, dx$$

$$= \int \cos^4 x\, (\sin x)^{-4} (\cos x\, dx)$$

let $u = \sin x$

$du = \cos x\, dx$

$\cos^4 x = (\cos^2 x)^2$

$= (1 - \sin^2 x)^2$

$$= \int (1 - \sin^2 x)^2 (\sin x)^{-4} (\cos x dx)$$

$$= \int (1-u^2)^2 u^{-4}\, du$$

$$= \int (1 - 2u^2 + u^4) u^{-4}\, du$$

$$= \int (u^{-4} - 2u^{-2} + 1)\, du$$

$$= \frac{u^{-3}}{-3} - 2\left(\frac{u^{-1}}{-1}\right) + u + C$$

$$= \frac{-1}{3}(\sin x)^{-3} + 2(\sin x)^{-1} + \sin x + C$$

$$= \frac{-1}{3\sin^3 x} + \frac{2}{\sin x} + \sin x + C$$

$$= \boxed{\frac{-1}{3}\csc^3 x + 2\csc x + \sin x + C}$$

EX 4 $\int \cos^2 x \sin^4 x \, dx$

> *Type 2*
> *If m or n is odd and positive, factor out* $\sin x$ *or* $\cos x$ *and use* $\sin^2 x + \cos^2 x = 1$.
> *If m and n are even and positive, use half – angle identities.*

$$\sin^2 x = \frac{1-\cos(2x)}{2} \qquad \cos^2 x = \frac{1+\cos(2x)}{2}$$

$$= \int \cos^2 x \sin^2 x \sin^2 x \, dx$$

$$= \int \left(\frac{1+\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right) dx$$

$$= \frac{1}{8}\int (1-\cos^2(2x))(1-\cos(2x))\, dx$$

$$= \frac{1}{8}\int \left(1-\cos(2x)-\cos^2(2x)+\cos^3(2x)\right) dx$$

$$= \frac{1}{8}\int (1-\cos(2x))\,dx - \frac{1}{8}\int \underbrace{\cos^2(2x)\,dx}_{\text{Type 1}} + \frac{1}{8}\int \underbrace{\cos^3(2x)\,dx}_{\text{Type 1}}$$

$$= \frac{1}{8}\left(x - \frac{\sin(2x)}{2}\right) - \frac{1}{8}\int \frac{1+\cos(4x)}{2}\,dx + \frac{1}{8}\int \cos^2(2x)(\cos 2x\,dx)$$

$$= \frac{1}{8}x - \frac{1}{16}\sin(2x) - \frac{1}{16}\int (1+\cos(4x))\,dx + \frac{1}{8}\int (1-\sin^2(2x))\cos(2x)\,dx$$

$$= \frac{1}{8}x - \frac{1}{16}\sin(2x) - \frac{1}{16}x - \frac{1}{16}\left(\frac{\sin(4x)}{4}\right)$$
$$+ \frac{1}{8}\int \cos(2x)\,dx - \frac{1}{8}\int \sin^2(2x)\cos(2x)\,dx$$

$$u = \sin(2x)$$
$$du = \cos(2x)(2)\,dx$$
$$\frac{1}{2}du = \cos(2x)\,dx$$

$$= \frac{1}{16}x - \frac{1}{16}\sin(2x) - \frac{1}{64}\sin(4x)$$
$$+ \frac{1}{8}\left(\frac{\sin(2x)}{2}\right) - \frac{1}{8}\left(\frac{1}{2}\right)\int u^2\,du$$

$$= \frac{1}{16}x - \cancel{\frac{1}{16}\sin(2x)} - \frac{1}{64}\sin(4x)$$
$$+ \cancel{\frac{1}{16}\sin(2x)} - \frac{1}{16}\left(\frac{u^3}{3}\right) + C$$

$$= \boxed{\frac{1}{16}x - \frac{1}{64}\sin(4x) - \frac{1}{48}(\sin^3(2x)) + C}$$

EX 5 $\int \sin(4x)\cos(5x)\,dx$

> *Type 3*
> *Use product identities:*
> (5) $\sin(mx)\cos(nx) = \frac{1}{2}[\sin((m+n)x) + \sin((m-n)x)]$
> (6) $\sin(mx)\sin(nx) = -\frac{1}{2}[\cos((m+n)x) - \cos((m-n)x)]$
> $\cos(mx)\cos(nx) = \frac{1}{2}[\cos((m+n)x) + \cos((m-n)x)]$

$= \int \frac{1}{2}\left[\sin(9x) + \sin(-x)\right] dx$

$= \frac{1}{2}\left[\frac{-\cos(9x)}{9} + \frac{-\cos(-x)}{-1}\right] + C$

$= \frac{-1}{18}\cos(9x) + \frac{1}{2}\cos(-x) + C = \boxed{\frac{-1}{18}\cos(9x) + \frac{1}{2}\cos x + C}$

EX 6 $\int_{-4}^{4} \underbrace{\sin\left(\frac{m\pi x}{4}\right)\sin\left(\frac{n\pi x}{4}\right)}_{\text{even fn}} dx$

$\boxed{A} = 2\int_0^4 \sin\left(\frac{m\pi x}{4}\right)\sin\left(\frac{n\pi x}{4}\right) dx$

Two cases: ① $m \neq n$
② $m = n$

<u>Case 1</u> $(m \neq n)$ $\boxed{A} = \frac{2}{2}\int_0^4 \left[\cos\left(\frac{(m+n)\pi x}{4}\right) - \cos\left(\frac{(m-n)\pi x}{4}\right)\right] dx$

$= \int_0^4 \left[\cos\left(\frac{(m+n)\pi}{4}x\right) - \cos\left(\frac{(m-n)\pi}{4}x\right)\right] dx$

$= \left(\frac{\sin\left(\frac{(m+n)\pi}{4}x\right)}{\frac{(m+n)\pi}{4}} - \frac{\sin\left(\frac{(m-n)\pi}{4}x\right)}{\frac{(m-n)\pi}{4}}\right)\Bigg|_0^4$

$= \left(\frac{\sin((m+n)\pi)}{\frac{(m+n)\pi}{4}} - \frac{\sin((m-n)\pi)}{\frac{(m-n)\pi}{4}}\right) - 0$ (because $\sin 0 = 0$)

note: if $m, n \in \mathbb{Z}$ (i.e. m, n both integers), then $\sin((m+n)\pi) = \sin((m-n)\pi) = 0$

$\boxed{A} = \boxed{\frac{4}{(m+n)\pi}\sin((m+n)\pi) - \frac{4}{(m-n)\pi}\sin((m-n)\pi)}$

<u>Case 2</u>: $m = n$

$\boxed{A} = 2\int_0^4 \sin^2\left(\frac{n\pi x}{4}\right) dx$ (need half angle identity)

$= 2\left(\frac{1}{2}\right)\int_0^4 \left(1 - \cos\left(\frac{n\pi x}{2}\right)\right) dx$

$= \left(x - \frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right)\Bigg|_0^4$

$= \left(4 - \frac{\sin(2n\pi)}{\frac{n\pi}{2}}\right) - (0)$

$= \boxed{4 - \frac{2\sin(2n\pi)}{n\pi}}$

<u>note</u>: if $n \in \mathbb{Z}$, then $\sin(2n\pi) = 0$

In Conclusion

integrals of trig. fns.

Type 1, Type 2, Type 3

all use ① Pythagorean identity

or ② Half-angle formulas

or ③ Product-to-Sum identities