

## Math 1060 ~ Trigonometry

### 23 The Dot Product

#### Learning Objectives

In this section you will:

- Find the dot product of two vectors.
- Learn properties of the dot product.
- Determine the angle between two vectors.
- Determine whether or not two vectors are orthogonal.
- Solve applications of the dot product.

$\sin^2 u + \cos^2 u = 1$

$\sin 2u = 2 \sin u \cos u$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab \cos C$

### Dot Product

The dot product of two vectors is a scalar. It can be useful in finding the angle between two vectors.

If  $\mathbf{v} = \langle v_1, v_2 \rangle$  and  $\mathbf{w} = \langle w_1, w_2 \rangle$ , then  $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$ .

*Note:*  $\vec{w} \cdot \vec{w} = w_1 w_1 + w_2 w_2 = w_1^2 + w_2^2 = \|\vec{w}\|^2$  i.e.  $\|\vec{w}\|^2 = \vec{w} \cdot \vec{w}$

Ex 1: Find the dot product of these pairs of vectors.

a)  $\mathbf{v} = \langle 3, 4 \rangle$  and  $\mathbf{w} = \langle -2, 5 \rangle$ .

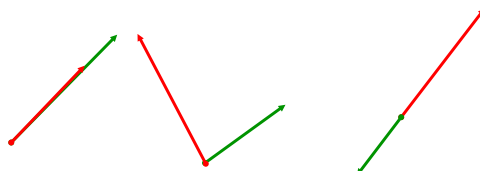
b)  $\mathbf{v} = \langle -3, 2 \rangle$  and  $\mathbf{w} = \langle -4, -6 \rangle$

### Geometric Interpretation of the Dot Product

same direction

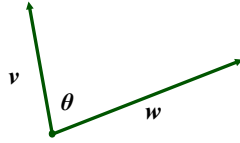
angle

opposite directions



*Note:* there are more dot product properties listed in the book to refer to.

We will use the Law of cosines to prove that  $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$ ,  $0 < \theta < \pi$ .



$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$

Ex 2: Determine the angle between these pairs of vectors.

a)  $\mathbf{v} = \langle 3, 4 \rangle$  and  $\mathbf{w} = \langle -2, 5 \rangle$ .

b)  $\mathbf{v} = \langle -3, 2 \rangle$  and  $\mathbf{w} = \langle -4, -6 \rangle$

Orthogonal vectors: If two vectors are perpendicular to each other they are said to be orthogonal. What would the cosine of the angle between two orthogonal vectors be?

Ex 3: Determine whether these pairs are vectors are orthogonal or not.

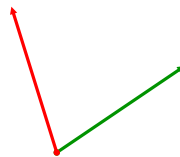
a)  $\langle 3, -2 \rangle$  and  $\langle 1, 4 \rangle$

b)  $\langle 4, -6 \rangle$  and  $\langle -3, -2 \rangle$

c)  $\langle 2, -1 \rangle$  and  $\langle -4, 2 \rangle$

### Orthogonal Projection

If  $\mathbf{v}$  and  $\mathbf{w}$  are nonzero vectors, then the orthogonal projection of  $\mathbf{v}$  onto  $\mathbf{w}$ , denoted by  $\text{proj}_{\mathbf{w}}(\mathbf{v})$  is given by


$$\begin{aligned} \text{proj}_{\mathbf{w}}(\mathbf{v}) &= \left( \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \right) \mathbf{w} = \left[ \mathbf{v} \cdot \left( \frac{\mathbf{w}}{\|\mathbf{w}\|} \right) \right] \left( \frac{\mathbf{w}}{\|\mathbf{w}\|} \right) \\ &= (\mathbf{v} \cdot \hat{\mathbf{w}}) \hat{\mathbf{w}} \end{aligned}$$

Ex 4: For  $\mathbf{v} = \langle -6, -5 \rangle$  and  $\mathbf{w} = \langle 10, -8 \rangle$ , find  $\text{proj}_{\mathbf{w}}(\mathbf{v})$ .

In physics, you will discover how this concept relates to problems about work.