Matrix Operations

In section 8.2 you will learn to:

- Decide whether two matrices are equal.
- Add and subtract matrices and multiply a matrix by a scalar.
- Multiply two matrices.
- Set up an $n \times n$ Identity matrix.

scalar is
a constant

Matrix addition: A + B

If $A = [a_{ij}]$ and $B = [b_{ij}]$, then $A+B = [a_{ij} + b_{ij}]$

A and B must be the same size as is the sum.
Each element in the sum is the sum of the corresponding elements.

3x2

$$A + B = \begin{bmatrix} 2 & 1 \\ 3 & -2 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 3 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 6 & 0 \\ -1 & 9 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \\ 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -2 \\ 3 & 2 \\ -1 & 4 \end{bmatrix}$$

Scalar Multiplication - A scalar is a real number or constant.

$cA = [ca_{ij}]$ ie. Every element of A gets multiplied by c.

Scalar → in lower case

matrix → in upper case

$$cA = \begin{bmatrix} 2c & c \\ 3c & -2c \\ 0 & 5c \end{bmatrix}$$

Properties of matrix multiplication:

- A + B = B + A — commutativity of addition
- → A + (B + C) = (A + B) + C — associativity of addition
- cd(A) = c(dA) — associativity w/ scalar mult.
- 1·A = A
- c(A+B) = cA + cB — distributivity w/ scalar mult.
- (c+d)A = cA + dA

Properties of matrix addition and scalar multiplication -

- A + B = B + A
- A + (B + C) = (A + B) + C
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3x3

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 3 & -2 & 1 \\ 0 & 5 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -2 & 1 \\ 3 & 2 & 5 \\ -1 & 4 & 0 \end{bmatrix}$$

Example 1

a) A + B =

$$\begin{bmatrix} 2 & -1 & -2 \\ 6 & 0 & 6 \\ -1 & 9 & -2 \end{bmatrix}$$

b) A - B =

$$\begin{bmatrix} 2 & 1 & -3 \\ 3 & -2 & 1 \\ 0 & 5 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -1 \\ -3 & -2 & -5 \\ 1 & -4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

c) 3A - 2B =

$$= \begin{bmatrix} 6 & 3 & -9 \\ 9 & -6 & 3 \\ 0 & 15 & -6 \end{bmatrix} + \begin{bmatrix} 0 & 4 & -2 \\ -6 & -4 & -10 \\ 2 & -8 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 7 & -11 \\ 3 & -10 & -7 \\ 2 & 7 & -6 \end{bmatrix}$$

Matrix Multiplication

$A = [a_{ij}]$ an $m \times n$ matrix $\quad$ $B = [b_{ij}]$ an $n \times p$ matrix

$AB = [c_{ij}]$ an $m \times p$ matrix

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{in}b_{nj}$

only do matrix multiplication when # columns of 1st matrix = # rows of 2nd matrix

$(m \times n)(n \times p) \rightarrow m \times p$

Example 2: Find AB if possible, then find BA

$A = \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix}$ $\quad$ $B = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$

$3 \times 2 \quad 2 \times 2 \rightarrow 3 \times 2$

$AB = \begin{bmatrix} -1(1)+3(0) & -1(2)+3(7) \\ 4(1)+-5(0) & 4(2)+-5(7) \\ 0(1)+2(0) & 0(2)+2(7) \end{bmatrix} = \begin{bmatrix} -1 & 19 \\ 4 & -27 \\ 0 & 14 \end{bmatrix}$

Notice: AB ≠ BA

$(2 \times 2)(2 \times 2) \rightarrow (2 \times 2)$

$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2(1)+-1(2) & 2(0)+-1(3) \\ 3(1)+4(2) & 3(0)+4(3) \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 11 & 12 \end{bmatrix}$

$\neq$

$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1(2)+0(3) & 1(-1)+0(4) \\ 2(2)+3(3) & 2(-1)+3(4) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 13 & 10 \end{bmatrix}$

The Identity matrix:

$$5 \cdot 1 = 1 \cdot 5 = 5$$

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$I = [1] \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* Must be square
* The diagonal has all 1s $I_{jj}=1$
* Zeros in all other positions

Notice that $I \cdot A = A \cdot I = A$

$$\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)+3(0) & 1(0)+3(1) \\ -2(1)+5(0) & -2(0)+5(1) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1(1)+0(-2) & 1(3)+0(5) \\ 0(1)+1(-2) & 0(3)+1(5) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$$

Matrix Multiplication Properties:

1. A(BC) = (AB)C associativity of mult.
2. A(B+C) = AB + AC
3. (A+B)C = AC + BC

distributivity

4. c(AB) = (cA)B = A(cB) commutativity of scalar mult. + associativity
5. I · A = A · I = A

Example 3: Find AB if possible

4×1 1×2 → 4×2

$$A = \begin{bmatrix} 3 \\ -1 \\ 5 \\ 4 \end{bmatrix} \qquad B = [5\ \ -3] + [0\ \ 2] + [4\ \ -1] = [9\ \ -2]$$

$$\begin{bmatrix} 3 \\ -1 \\ 5 \\ 4 \end{bmatrix}\begin{bmatrix} 9 & -2 \end{bmatrix} = \begin{bmatrix} 27 & 3(-2) \\ -1(9) & -1(-2) \\ 5(9) & 5(-2) \\ 4(9) & 4(-2) \end{bmatrix} = \begin{bmatrix} 27 & -6 \\ -9 & 2 \\ 45 & -10 \\ 36 & -8 \end{bmatrix}$$

Example 4: Find AB, BA, and A^2, if possible

2x2 2x2 $A^2 = AA$

$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1(1)+-1(-3) & 1(3)+-1(1) \\ 2(1)+1(-3) & 2(3)+1(1) \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & 7 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)+3(2) & 1(-1)+3(1) \\ -3(1)+1(2) & -3(-1)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ -1 & 4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)+-1(2) & 1(-1)+-1(1) \\ 2(1)+1(2) & 2(-1)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 4 & -1 \end{bmatrix}$$

Put this system into matrix algebra form: A·X = C
A is the Matrix of coefficients.
X is the matrix of variables.
C is the matrix of constants.

$$\begin{aligned} x - y + 4z &= 17 \\ x + 3y \quad &= -11 \\ 2y + 5z &= 0 \end{aligned}$$

$$A \cdot X = C$$

$$\begin{bmatrix} 1 & -1 & 4 \\ 1 & 3 & 0 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ -11 \\ 0 \end{bmatrix}$$