UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Algebraic Topology
May 25, 2023

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points.

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1. Prove that if X and Y are path connected spaces then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

2. Let Σ_2 be the closed, orientable genus two surface. Show that $\pi_1(\Sigma_2, x)$ has a normal subgroup G with the quotient group $\pi_1(\Sigma_2)/G$ isomorphic to $\mathbb{Z}$. You can use a picture as part of your proof.

3. Let Δ^n be the n-simplex with its usual Δ-complex structure. Find all subcomplexes X of Δ^n such that $\tilde{H}_{n-1}(X)$ is non-zero.

4. Let $X = S^1 \times [0,1]$ and $\partial X = S^1 \times \{0,1\}$. Calculate $H_i(X, \partial X)$ for all i.

5. (a) Construct a single chain complex C such that C has the following three properties:
   - $H_i(C)$ is non-zero if and only $i = 1$.
   - There is a coefficient group G_0 such that $\tilde{H}^i(C; G_0) = 0$ for all i.
   - There is a coefficient group G_1 such that $\tilde{H}^i(C; G_1)$ is non-zero if and only if $i = 1, 2$.

   (b) Find a topological space X such that $H_i(C) = \tilde{H}_i(X)$ for all i.

6. Show that $S^2 \vee S^4$ is not homotopy equivalent to a compact manifold without boundary.