Qualifying Exam
Analysis of Numerical Methods I, January 2024

Instructions: This exam is closed book, no notes, and no electronic devices or calculators are allowed. You have two hours and you will be graded on work for only 3 out of the 4 questions below. All questions have equal weight and a cumulative score of 65% or more on your 3 graded questions is considered a pass. A cumulative score of 80% or greater is considered a high pass. Indicate clearly the work and the 3 questions that you wish to be graded.

Problem 1.(**QR and Cholesky Factorizations**). Let A be a nonsingular square matrix and let $A = QR$ and $A^*A = U^*U$ be the QR factorization of A and the Cholesky factorization of A^*A, respectively. Assume that the usual normalizations $r_{jj}, u_{jj} > 0$ are in effect. Is it true or false that $R = U$? Justify your answer with mathematical details.

Problem 2. (**Properties via SVD**).
Suppose $A \in \mathbb{C}^{m\times m}$ has an SVD, $A = U\Sigma V^*$. Find an eigenvalue decomposition of the $2m \times 2m$ hermitian matrix (using information about SVD for A),

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}.$$

Problem 3. (**Properties of Projectors**).
Prove that a projector P is orthogonal if and only if $P = P^*$.

Problem 4. (**Properties of Gaussian Elimination**).
Show that for Gaussian elimination with partial pivoting applied to any matrix $B \in \mathbb{C}^{n\times n}$, the growth factor ρ satisfies $\rho \leq 2^{n-1}$. Recall that the growth factor is defined as,

$$\rho = \frac{\max_{i,j} |u_{ij}|}{\max_{i,j} |b_{ij}|},$$

where $u_{i,j}$ are the entries of the upper triangular matrix that results from performing Gaussian elimination on B.