UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Preliminary Qualifying Exam for Math 6410, Ordinary Differential Equations,

January 6, 2023.

Instructions: This examination has six problems of which you are expected to work four. If you work more than the four required number of problems, then state which problems you wish to have graded. In order to receive maximum credit, solutions to problems must be clearly and carefully presented. All problems are worth 25 points each.

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1. (a) What is the statement of Gronwall's inequality? Where to start: Suppose $\psi(x)$ satisfies the inequality

$$\psi(t) \leq \alpha + \int_0^t \beta\psi(s)ds,$$

then $\cdots$.

(b) Suppose $f(t,x) \in C(U, \mathcal{R}^n)$ is Lipschitz continuous in its second argument, and let $x(t)$ and $y(t)$ be solutions of the same differential equation $\frac{dx}{dx} = f(t,x)$ with different initial data $x(0) = x_0$, $y(0) = y_0$. Prove that

$$|x(t) - y(t)| \leq |x_0 - y_0| \exp(Lt),$$

for some positive constant L and some positive time $0 \leq t < T$. What is an estimate for the constant L?

(c) Why may the inequality fail to hold for all time? Give an example of where this fails.

2. (a) Suppose the function $f(t,x)$ is periodic in t, $f(t+1,x) = f(t,x)$. Define the Poincare map for the differential equation $\dot{x} = f(t,x)$.

(b) Consider the differential equation

$$\dot{x} = x(1-x) - h(t)x,$$

where $h(t) \geq 0$ is periodic, $h(t+1) = h(t)$. Under what conditions on $h(t)$ does this equation have a strictly positive periodic solution?

(c) Suppose $h(t) = a\sin^2(\pi x)$. What condition on a guarantees the existence of a positive periodic solution?

3. Consider the differential equation $\ddot{x} + \eta\dot{x} + x^3 - x = 0$, with $0 < \eta < 2\sqrt{2}$ a constant.

(a) Sketch a phase portrait for this equation. Identify all fixed points and their type.

(b) Find a first integral of the motion if $\eta = 0$. Make a sketch of these integral curves in the $x - \dot{x}$ plane.

(c) Suppose that at time $t = 0$, $\frac{\dot{x}^2}{2} + \frac{1}{4}x^4 - \frac{1}{2}x^2 < 0$, and $0 < x < \sqrt{2}$. Determine $\lim_{t\to\infty} x(t)$, and explain how to prove this.

4. For a system of equations $\frac{dx}{dt} = A(t)x$, where $x \in \mathbb{R}^n$ and $A(t)$ is a real $n \times n$ matrix function which is smooth in t,

(a) What is Abel's formula? (State without proof).

(b) Suppose $A(t)$ is periodic, $A(t+T) = A(t)$, $T > 0$. Define the monodromy matrix M.

(c) Show that the eigenvalues μ_i, $i = 1, \cdots, n$ of the monodromy matrix satisfy $\Pi_{i=1}^n \mu_i = \exp\Big(\int_0^T trace(A(t))dt\Big)$.

(d) Suppose $|\Pi_{i=1}^n \mu_i| > 1$. Prove that the zero solution of $\frac{dx}{dt} = A(t)x$ is unstable. If $|\Pi_{i=1}^n \mu_i| < 1$, does that guarantee that the zero solution of $\frac{dx}{dt} = A(t)x$ is stable? Why or why not?

(e) Suppose

$$A(t) = \begin{pmatrix} \lambda(t) & \omega(t) \\ -\omega(t) & \lambda(t) \end{pmatrix},$$

where both $\lambda(t)$ and $\omega(t)$ are periodic functions of t with period $T = 1$. Find a condition on $\lambda(t)$ that determines stability or instability of the zero solution of $\frac{dx}{dt} = A(t)x$. Hint: Examine $\frac{d}{dt}(x_1^2 + x_2^2)$.

5. Consider the system of equations

$$\frac{dx}{dt} = \cos^2(x)\Big(y\cos(x) - \sin(x)\Big), \quad \frac{dy}{dt} = -\sin(x) - y\cos(x),$$

with $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

(a) Sketch a phase portrait for this system of equations. Identify all fixed points and their stability.

(b) Show that the rectangle with vertices (x_0, y_0), $(-x_0, y_0)$, $(-x_0, -y_0)$, $(x_0, -y_0)$, where $y_0 = \tan(x_0)$, is positively invariant for any x_0, $0 < x_0 < \frac{\pi}{2}$.

(c) What is $\omega_+(x, y)$ and for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Are there any periodic orbits?

(d) What is $\omega_+(x, y)$ for $x = \frac{\pi}{2}$? What is $\omega_+(x, y)$ for $x = -\frac{\pi}{2}$.

6. Consider the differential operator $Lu = -u''$ with boundary conditions $u'(0) = u'(1) = 0$.

(a) What Hilbert space and inner product is used for the analysis of this operator?

(b) Is this operator symmetric? Verify your answer.

(c) Is this operator invertible? Why or why not?

(d) What are the eigenvalues and eigenfunctions for this operator?

(e) Are these eigenfunctions complete on the domain of the operator? If yes, how do you know (what theorem can you invoke?) and if no, produce a function in the domain of L that is orthogonal to all of the eigenfunctions.

(f) Is $f(x) = \cos^2(\pi x)$ in the domain of the operator? Find a representation of $f(x)$ in terms of the eigenfunctions of L, if it exists?

(g) Is $f(x) = \cos^2(\pi x)$ in the range of L? Why or why not?