UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. preliminary Examination on
Applied Linear Operators and Spectral Methods (Math 6710)
January 2022

Instructions: This examination includes five problems but you are to work three of them. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three problems will be graded. All problems are worth 20 points. A pass is 35 or more points. A high-pass is 45 or more points.

1. Assume a bounded, self-adjoint operator $T : H \to H$, where H is a Hilbert space, satisfies
$$\langle Tx, x \rangle \geq \beta \|x\|^2, \tag{1}$$
for some $\beta > 0$. The goal of this problem is to show that T is one-to-one and onto.
   (a) Show that the nullspace of T is trivial, i.e. $\mathcal{N}(T) = \{0\}$.
   (b) Show that the range $\mathcal{R}(T)$ is closed.
   (c) Show that $\mathcal{R}(T)^\perp = \{0\}$.
   (d) Why do we have $\mathcal{R}(T) = H$?

2. Let $T : H \to H$ be a bounded linear operator on a Hilbert space H. Show that T is compact if and only if T^*T is compact.

3. Let H be a separable Hilbert space with (e_n) being a total orthonormal sequence of H. Let $T : H \to H$ be a bounded linear operator satisfying
$$\sum_{n=1}^{\infty} \|Te_n\| < \infty.$$
Show that T is compact by approximating T by a sequence (T_k) of finite rank operators that converges to T in an appropriate sense.

4. Let H be a complex Hilbert space and let $T \in B(H, H)$. The goal of this problem is to prove that $\|T\|^2 = \|T^*T\|$.
   (a) Show that $\|T\|^2 \leq \|T^*T\|$.
   (b) Show that $\|T^*T\| \leq \|T\|^2$.

5. Let X be a normed vector space. Let $x_0 \in X$ and consider the closed ball $\widetilde{B}(0, \|x_0\|)$. Use the Hahn-Banach theorem to show that there exists $f \in X'$ such that $f(x_0) = 1$ and for any $x \in \widetilde{B}(0, \|x_0\|)$ we have $f(x) \leq 1$. **Do not use Hahn-Banach for convex sets or the separating hyperplane theorem**.