UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Real Analysis
January, 2021.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points.

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1. Let μ be the outer measure on $\mathbb{R}$ defined by

$$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) \mid A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i],\ a_i \leq b_i \right\}$$

for every $A \subset \mathbb{R}$. Prove that $\mu((a, b]) = b - a$. You may use that the length of a segment covered by finitely many segments is less than or equal to the sum of the lengths of segments in the covering.

2. (a) State the Monotone Convergence Theorem.

   (b) Let $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = e^{-x}$. Explain why f is Lebesgue integrable and compute its integral. You may use the fact that if a function $f : [0, \infty) \to \mathbb{R}$ is 0 outside an interval $[a, b]$ and $f|[a, b] : [a, b] \to \mathbb{R}$ is Riemann integrable then f is Lebesgue integrable and the Lebesgue integral of f equals the Riemann integral of $f|[a, b]$.

3. Let a measure space $(X, \mathcal{M}, \mu)$ be given and for a fixed measurable function $f : X \to [0, \infty)$ define a new measure λ on $\mathcal{M}$ by

$$\lambda(E) = \int_E f \, d\mu$$

You don't have to prove that λ is a measure. Prove that for any measurable $g : X \to [0, \infty)$

$$\int g \, d\lambda = \int gf \, d\mu$$

4. (a) State the Closed Graph Theorem.

   (b) Let $T : U \to V$ be a linear map between two Banach spaces such that, for any bounded linear functional f on V, the composite $f \circ T$ is a bounded functional on U. Prove that T is bounded. You may use the fact that bounded linear functionals on a Banach space separate points.

5. (a) State the Baire Category Theorem.

   (b) Let V be a Banach space, and $T : V \to V$ a bounded linear map. Assume that for every $v \in V$ there exists a non-negative integer n such that $T^n(v) = 0$. Prove that there exists an integer n such that $T^n(v) = 0$ for all $v \in V$.

6. (a) Give the definition of what it means for a sequence (x_n) in a Banach space X to converge *weakly* to $x \in X$.

   (b) Let $X = [0, 1]$ equipped with the Lebesgue measure. For $n = 1, 2, \ldots$, let $f_n = n \cdot \chi_{[0,1/n]}$, where χ_A denotes the indicator function for the set A. Prove that the sequence f_n does not converge weakly to 0 in $L^1([0, 1)]$.